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Regularity of solutions and the free boundary for a class of Bernoulli-type parabolic free boundary problems with variable coefficients

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REGULARITY OF SOLUTIONS AND THE FREE BOUNDARY FOR A CLASS OF BERNOULLI-TYPE PARABOLIC FREE BOUNDARY PROBLEMS WITH VARIABLE COEFFICIENTS

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For my parents.

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ABSTRACT

Backing, Thomas H. PhD, Purdue University, May 2016. Regularity of Solutions and the Free Boundary for a class of Bernoulli-type Parabolic Free Boundary Problems with Variable Coefficients. Major Professor: Donatella Danielli.

In this work the regularity of solutions and of the free boundary for a type of parabolic free boundary problem with variable coefficients is proved. After introducing the problem and its history in the introduction, we proceed in Chapter 2 to prove the optimal Lipschitz regularity of viscosity solutions under the main assumption that the free boundary is Lipschitz. In Chapter 3, we prove that Lipschitz free boundaries possess a classical normal in both space and time at each point and that this normal varies with a Hölder modulus of continuity. As a consequence, the viscosity solution is in fact a classical solution to the problem.

1. Introduction

1.1 Statement of the Problem and Main Result

In this thesis, we study the regularity of viscosity solutions and the regularity of the free boundary for a family of parabolic free boundary problems of the form

$$\begin{cases} \mathcal{L}u - u_t = 0 \quad \text{in } (\{u > 0\} \cup \{u \leq 0\}^\circ) \subset \Omega \\ G(u_\nu^+, u_\nu^-) = 1 \quad \text{along } \partial\{u > 0\} \subset \Omega. \end{cases} \tag{1.1}$$

A typical choice of the free boundary condition in the second line is $G(u_\nu^+, u_\nu^-) = (u_\nu^+)^2 - (u_\nu^-)^2$, but this work examines more general boundary conditions for which this condition serves as a prototype. Here

$$\mathcal{L} = \sum a_{ij}(x,t) D_{ij}$$

is a uniformly elliptic operator with Hölder continuous coefficients. The set $\partial\{u > 0\}$ is the free boundary and ν is the normal to this surface; u_ν^+ and u_ν^- are the directional derivatives of u taken from the positive side of the free boundary and the negative side, respectively. Finally, $\{u \leq 0\}^\circ$ denotes the interior of the set $\{u \leq 0\}$. The overdetermined condition on the free boundary consists in imposing the normal derivative constraint appearing in (1.1), in addition to requesting the natural continuity condition $u = 0$.

As is typical in a free boundary problem the free boundary is not specified initially so we do not know *a priori* that the free boundary possesses a classical normal at each point or that u is continuously differentiable up to the boundary. As a consequence the concept of a solution to (1.1) requires explanation. In this work the concept of a solution is that of a viscosity solution, which is essentially a solution defined by how it compares with classical solutions to the problem. One of the main questions in the theory is determining conditions under which a viscosity solution to (1.1) is a classical solution. Clearly the free boundary $\partial\{u > 0\}$ needs to be a C^1 surface for u to solve (1.1) in a classical sense. Hence the question becomes under what conditions the free boundary is a differentiable surface.

In the theory of free boundary problems we cannot, in general, prove that a weak solution u is a classical solution without imposing additional assumptions. In particular, the free boundary may not be smooth enough to allow a classical interpretation of of the free boundary condition. As an example, it is known that minimizers of the Alt-Caffarelli-Friedman energy functional, a functional which has a connection to (1.1) outlined below, may not have smooth free boundaries if the dimension is large enough. The free boundary may exhibit corners, and at such point a the solution u cannot be a classical solution. For this reason it is typical in free boundary problems to assume some additional structure or weak regularity on the free boundary, then push this to higher regularity of the free boundary or the solution.

This interaction between results regarding the regularity of the solution and the free boundary is typical in a free boundary problems. In this work this interaction takes the form of assuming Lipschitz regularity of the free boundary, which enables us to prove the optimal Lipschitz regularity of solutions. Then, using both the assumed Lipschitz regularity of the free boundary and the consequential Lipschitz regularity of the solution, higher regularity of the free boundary is proved. This higher regularity in turn implies by a standard, if non-trivial, argument that the solution u takes up the boundary condition with continuity and is therefore a classical solution to our problem.

1.2 History and Context of the Problem

The free boundary problem (1.1) has a long history that begins with the elliptic version of the equation. The free boundary problem in (1.1) is formally the Euler-Lagrange equation for the Alt-Caffarelli-Friedman energy minimization problem studied in [AC] and [ACF]. In the simplest one-phase formulation of the problem (that is, when the solution is assumed to be non-negative), it consists of studying the minimizers, subject to some prescribed boundary data, of the functional

$$J(u) = \int_\Omega |\nabla u|^2 + \chi_{\{u>0\}} \, \mathrm{d}x.$$

The two phase version of this functional has $\lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}}$; this yields a stationary version of the free boundary problem (1.1) with $a_{ij} = \delta_{ij}$. The jump in the derivative for the corresponding Euler-Lagrange equation occurs because of the different weights assigned to the positive and negative phases in the functional. Minimizers of the energy will solve the equation in a weak measure-theoretic sense, but the main tool in the analysis of this problem is exploiting the minimization property. This fact allows better results to be obtained for minimizers than mere weak solutions, chief among them the result that in certain dimensions the minimizer is a classical solution to the equation. This result is stronger than the sort of result that can be proved for weak solutions; in general it is not possible to prove that a weak solution is classical without some additional assumptions.

The problem (1.1) also has a connection to combustion theory. From this perspective the parabolic version of the problem arises as the limit of a singular perturbation problem which models combustion. The flame front corresponds to the free boundary of the solution u; it is time dependent due to the flame front advancing through a flammable medium over time. The setup consists of studying the limit as $\varepsilon \to 0$ of solutions to

$$\Delta u^\varepsilon - u_t^\varepsilon = \beta_\varepsilon(u^\varepsilon)$$

where $\beta(s)$ is a Lipschitz function supported on $[0, 1]$ with

$$\int_0^1 \beta(s) \, \mathrm{d}s = M \quad \text{and} \quad \beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon}).$$

Under the assumption that $u^\varepsilon \geq 0$, it was shown in [CV] that the boundary condition for the limit function u is $u_\nu^+ = 1$. In [CLW1], [CLW2] the two phase version of this problem was studied and the free boundary condition for the limit solution was demonstrated to be $(u_\nu^+)^2 - (u_\nu^-)^2 = 2M$, M some positive constant. In both cases this free boundary condition holds in a suitable weak sense, in this case in terms of an asymptotic expansion of the function at 'regular' points along the free boundary. Combustion theory also offers the simplest explanation of the non-degeneracy hypothesis (H2) stated in the following section. In general a flame front might extinguish itself or two or more distinct flame fronts might merge. When this happens no real regularity of the free boundary can be expected. The non-degeneracy condition can then be interpreted as restricting our attention to the regions where regularity of the free boundary can be pushed to $C^{1,\alpha}$ by excluding degenerate flame front behavior.

A different perspective on the problem is found in the pioneering work of Caffarelli in [C1], [C2]. Here the perspective is to study the regularity of viscosity solutions and their free boundaries to the stationary version of (1.1) with $a_{ij} = \delta_{ij}$, i.e. Laplace's equation. It is in these works that the main ideas used in this paper, such as monotonicity cones, viscosity solutions to (1.1), and 'sup-convolutions' were first developed. These papers also provide the prototype results for this thesis; the regularity of Lipschitz free boundaries is treated in [C1] while the 'flat' case was treated in [C2]. Later these ideas were extended in [ACS1], [ACS2], [ACS3] to the parabolic Stefan problem, which models solid-liquid melting and solidification. Finally, the constant coefficient version of (1.1) was studied in [F]. All of these works, including the singular perturbation problem, involve only the case where $\mathcal{L} = \Delta$, Δ denoting the Laplacian.

When $\mathcal{L} = \Delta$, whether in the elliptic or parabolic case, extensive use is made of the fact that directional derivatives of solutions to a constant coefficient linear PDE are themselves solutions to the same PDE. In particular, tools like the Harnack Inequality can be applied to the directional derivatives. At variance with the case when $\mathcal{L} = \Delta$, directional derivatives of solutions to the operator $\mathcal{L} - \partial_t$ are not themselves solutions. This prevents a straightforward extension of the constant coefficient results to this variable coefficient case. Indeed, the only results extending these methods to a variable coefficient parabolic problem are the recent papers [FS1], [FS2] for the Stefan problem.

1.3 Precise Statement of Problem and Results

In this section we precisely lay out the problem and main results. We begin with a series of definitions.

We will denote the positivity set of u by Ω^+, i.e. $\Omega^+ = \{x \in \Omega \ |u(x) > 0)\}$; likewise the negative set is denoted by Ω^-. Occasionally we will write $\Omega^{\pm}(u)$ to emphasize the dependence of these domains on the function u. The set $\partial\{u > 0\}$ is the free boundary and will be denoted by $FB(u)$ or just FB.

We will denote by $C_{R,T}(x_0, t_0)$ the cylinder

$$B'_R(x_0) \times (t_0 - T, t_0 + T).$$

If the center of the cylinder is the origin we will simply write $C_{R,T}$ and if $R = T$ we will write C_R.

The operator

$$\mathcal{L} = \sum_{i,j} a_{ij}(x,t) D_{ij}$$

has Holder continuous coefficients $a_{ij} \in C^{0,\alpha}(\Omega)$, $0 < \alpha \leq 1$ and there exists $\lambda, \Lambda > 0$ such that

$$\lambda|\xi|^2 \leq \sum a_{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for all $(x,t) \in \Omega$. Denoting by $A(x,t)$ the matrix $[a_{ij}(x,t)]$, we assume $A(0,0) = [\delta_{ij}]$, i.e. the identity.

On $G(a,b)$ we will require:

1. G Lipschitz with constant L_G in both variables.

2. $G(a_1, b) - G(a_2, b) > c^*(a_1 - a_2)^p$ if $a_1 > a_2$ (strictly increasing in the first variable)

3. $G(a, b_1) - G(a, b_2) < -c^*(b_1 - b_2)^p$ if $b_1 > b_2$ (strictly decreasing in the second variable)

The p appearing here is some positive power.

Definition 1 *(Classical Subsolution/Supersolution) We say $v(x,t)$ is a classical subsolution (supersolution) to* (1.1) *if $v \in C^1(\overline{\Omega^+(v)}) \cup C^1(\overline{\Omega^-(v)})$, $\mathcal{L}v - v_t \geq 0$ ($\mathcal{L}v - v_t \leq 0$) in $\Omega^\pm(v)$ and*

$$G(v_\nu^+, v_\nu^-) \geq 1 \ \ (G(v_\nu^+, v_\nu^-) \leq 1), \quad \textit{where } \nu = \frac{\nabla v^+}{|\nabla v^+|}$$

A strict subsolution (supersolution) satisfies the above with strict inequalities.

Definition 2 *(Viscosity Subolutions/Supersolutions) A continuous function $v(x,t)$ is a viscosity subsolution (supersolution) to* (1.1) *in Ω if for every space-time cylinder $Q = B_r' \times (-T, T) \Subset \Omega$ and for every classical supersolution (subsolution) w in Q, the inequality $v \leq w$ ($v \geq w$) on $\partial_p Q$ implies that $v \leq w$ ($v \geq w$). Additionally, if w is a strict classical supersolution (subsolution), then $v < w$ ($v > w$) on $\partial_p Q$ implies $v < w$ ($v > w$) inside Q.*

We now turn to the hypotheses on the free boundary of u. Our main result will require that this free boundary is Lipschitz, but will we will also require a non-degeneracy condition to hold at regular points. We first define such points.

Definition 3 *(Regular Points) A point (x_0, t_0) on the free boundary of u is a right regular point if there exists a space-time ball $B_R \subset \Omega^+$ such that $B_R \cap \partial\{u > 0\} = \{(x_0, t_0)\}$.*

A point (x_0, t_0) on the free boundary of u is a left regular point if there exists a space-time ball $B_R \subset \Omega^-$ such that $B_R \cap \partial\{u \leq 0\} = \{(x_0, t_0)\}$.

We now state precisely our assumptions:

(H1) *The free boundary $FB(u)$ is the graph of a Lipschitz function f, that is, $FB(u) =$*

$\{(x', x_n, t)|f(x', t) = x_n\}$ *with* $f(0,0) = 0$. *We will denote by* L *and* L_0 *the Lipschitz constant of* f *in space and time respectively.*

(H2) *u satisfies the following non-degeneracy condition: There exists a* $m > 0$ *such that if* (x_0, t_0) *is a right regular point for* u *then*

$$\frac{1}{|B_r'(x_0)|}\int_{B_r'(x_0)} u^+ dx \geq mr. \tag{1.2}$$

Finally we can state the main results of this thesis:

Theorem 1 *Let* u *be a viscosity solution to* (1.1) *in* C_1 *satisfying (H1) and (H2). Then in* $C_{1/2}$, u *possesses a cone of monotonicity in both space and time and is Lipschitz continuous.*

Theorem 2 *Let* u *be a solution to our free boundary problem in* Q_1 *satisfying the hypotheses of this section. Then for every point* (x, t) *on the free boundary in* $Q_{1/2}$ *there exists a normal vector to the surface* $\eta(x, t)$. *Furthermore, this normal vector satisfies*

1. $|\eta(x,t) - \eta(y,t)| \leq C|x - y|^\alpha$

2. $|\eta(x,s) - \eta(x,t)| \leq C|s - t|^\beta$

Finally, the free boundary condition is taken up with continuity by the solution u *so that* u *is a classical solution to* (1.1).

1.4 Structure of the Work

This work is organized into two main parts. Proving the optimal Lipschitz regularity of the solution u is the focus of Chapter 2, whereas the regularity of the free boundary is the focus of Chapter 3. Each chapter begins with a precise statement of the problem under consideration, and a collection of known results and tools which are needed in the work; for the reader's convenience there is some redundancy in these preambles in order.

Chapter 2 is organized as follows: In the first section we briefly restate the main theorem to be proved. Section 2 collects the main tools and known results used the analysis of this problem. Section 3 contains our results on the asymptotic behavior of solutions near the free boundary. These results are used in Section 4 to prove that a space-time cone of monotonicity exists up to the free boundary for u. In Section 5, we use this to prove the Lipschitz regularity of u.

Chapter 3 is organized as follows: In the first section we reiterate our main result. In Section 2 we have collected the main tools and known results that we will need in our analysis. Section 2 deals with the interior enlargement of the monotonicity cone while Section 3 contains results that propagate a portion of this enlargement to the free boundary. Finally Section 4 contains the iteration used to prove the regularity of the free boundary in space while Section 5 contains a similar iteration used to prove the regularity in space-time.

2. Regularity of the Solution

2.1 Main Result of this Chapter

The main result of Volume 2 is the following theorem.

Theorem 1 *Let u be a viscosity solution to* (1.1) *in C_1 satisfying (H1) and (H2). Then in $C_{1/2}$, u possesses a cone of monotonicity in both space and time and is Lipschitz continuous.*

2.2 Main Tools

In this section we collect some of the essential tools and known results used in the analysis of (1.1).

Let

$$\Omega_{2r} = \{(x', x_n, t) : |x'| < 2L^{-1}r, |t| < 4L_0^{-2}r^2, f(x', t) < x_n < 4r\}.$$

Denote by $P_r = (0, r, 0)$, $\overline{P_r} = (0, r, 2L_0^{-2}r^2)$, $\underline{P_r} = (0, r, -2L_0^{-2}r^2)$. These are the inward point, forward point and backward point, respectively.

Denote by $\delta(X, Y)$ the parabolic distance between $X = (x, t)$ and $Y = (y, s)$, that is, $\delta(X, Y) = |x-y| + |t-s|^{1/2}$ and by δ_X the parabolic distance from X to the origin.

Our tools, valid for $\mathcal{L}$-caloric functions on Lipschitz domains vanishing on a piece of the boundary, are as follows (see [FS1], [FS2]):

Interior Harnack Inequality: There exists a positive constant $c = c(n, \lambda, \Lambda)$ such that for any $r \in (0, 1)$

$$u(\underline{P_r}) \leq cu(\overline{P_r}).$$

Carleson Estimate: There exists a $c = c(n, \lambda, \Lambda, L, L_0)$ and $\beta = \beta(n, \lambda, \Lambda, L, L_0)$, $0 < \beta \leq 1$ such that for every $X \in \Omega_{r/2}$

$$u(X) \leq c \left(\frac{\delta_X}{r} \right)^{\beta} u(\overline{P_r}).$$

Boundary Harnack Principle: There exists $c = c(n, \lambda, \Lambda, L, L_0)$ and $\beta = \beta(n, \lambda, \Lambda, L, L_0)$, $0 < \beta \leq 1$, such that for every $(x, t) \in \Omega_{2r}$ and u and v are two solutions

$$\frac{u(x,t)}{v(x,t)} \geq c \frac{u(\underline{P_r})}{v(\bar{P}_r)}.$$

Backward Harnack Inequality: Let $m = u(\underline{P_{3/2}})$ and $M = \sup_{\Omega_2} u$. Then there exists a positive constant $c = c(n, \lambda, \Lambda, L, L_0, M/m)$ such that if $r \leq 1/2$

$$u(\overline{P_r}) \leq c u(\underline{P_r}).$$

We will use c to denote constants which depend on some or all of n, λ, Λ, L, L_0, M/m. We will write $\Gamma(\theta, \eta)$ to denote a cone of directions with axis η and opening θ.

Definition 4 *(Monotonicity) A function $u \geq 0$ is ε_0-monotone in a domain Ω in a cone of directions $\Gamma(\theta, e_n)$ if there exists a $\bar{\beta} > 0$ such that for any $\varepsilon \geq \varepsilon_0$, $\tau \in \Gamma(\theta, e_n)$*

$$u(p) - u(p - \varepsilon\tau) \geq c\varepsilon^{\bar{\beta}} u(p)$$

provided both p and $p - \varepsilon\tau$ belong to Ω. A function $u \geq 0$ is fully monotone in the direction τ if for any $\varepsilon > 0$

$$u(p) - u(p - \varepsilon\tau) \geq 0.$$

A function u of arbitrary sign is ε_0-monotone if both u^+ and $-u^-$ are ε_0-monotone.

The following lemma, derived from Lemma 2.3 in [FS1], provides a starting point for our problem.

Lemma 1 *Let u be a viscosity solution to* (1.1)*. Then there exists a cone of directions $\Gamma(\theta, e_n)$ in which u is fully monotone in space and ε_0-monotone in time.*

Remark: By a rescaling argument we may assume that u is fully monotone in both space and time in the cone $\Gamma(\theta, e_n)$ outside an ε_0 neighborhood of FB(u) (see remarks at beginning of section 4 in [FS1]).

We list some technical constants used in the iteration: Let ε_0 be a small fixed number which is the ε_0-monotonicity. Let β, δ, γ be positive numbers such that

$$0 < \gamma = \frac{1-\delta}{2}, \quad 0 < \beta < \min\left\{\frac{1-\delta}{2}, \frac{\alpha+\delta-1}{2}\right\}.$$

Here α is the Holder exponent of the coefficients and δ will be the defect angle of the cone of monotonicity ($\delta = \pi/2 - \theta$).

Later we will show at each step of an iteration ε-monotonicity with a suitable $\varepsilon < \varepsilon_0$ and $\bar{\beta} = 1 - \gamma + \beta$. But first we quote two results from [FS1] that find application to this problem. Although the problem in [FS1] is the Stefan problem, these results only depend on the Lipschitz nature of the domain, the fact that u vanishes along the graph of f and that u is a $\mathcal{L}$-caloric function. Therefore they are valid in our case.

Lemma 2 *(Lemma 2.4 in [FS1])*

Let $\alpha \leq 1$ be the Holder exponent of the $a_{ij}(x,t)$ and let β, δ, γ be chosen as above. Suppose $u \geq 0$ is monotone in the e_n direction and

$$u(p) - u(p - \varepsilon\tau) \geq c\varepsilon^{1-\gamma+\beta}u(p) \geq c\varepsilon^{\frac{1+\delta}{2}+\beta}u(p)$$

for $d_p < \eta/4$ (d_p is the distance from $p = (x,t)$ to the free boundary at time level t) where $\tau = \beta_1 e_n + \beta_2 e_t$ with $\beta_1 > 0$, $|\beta_2| \neq 0$ and $|\tau| = 1$. Then if $M = M(n, L)$ is large enough and ε is small enough outside a $(M\varepsilon)^\gamma$ neighborhood of $F(u)$ we have

$$D_{\tau_\varepsilon} u \geq 0$$

where $\tau_\varepsilon = \tau + c(M\varepsilon)^{(\alpha+\delta-1)/2}e_n$ for some $c = c(n, L, L_0, \beta_1, \beta_2)$

Remark: The thrust of the lemma is that a spacially monotone solution u which is ε monotone in a space-time direction τ as above will be fully monotone in a slightly different space-time direction τ_ε if far enough from the free boundary. Note that if the original τ came from some cone of ε-monotonicity then u will be fully monotone in a smaller cone of directions away from the free boundary. The term $c(M\varepsilon)^{(\alpha+\delta-1)/2}e_n$ is describes the amount of the cone that must be given up to obtain full monotonicity. In what follows, the main idea will be an iteration that gives up a certain amount of the cone at each step in order to reduce the ε-monotonicity at each step, ultimately proving a cone of full monotonicity exists for u. We explicitly observe that, by rescaling, we may assume that Lemma holds 2 in Q_1.

The other ingredient our work will require is the following family of functions. We begin with the domains involved. Let

$$\mathcal{N}_{b\varepsilon} = \{p = (x', x_n, t) : d(p, FB(u)) < b\varepsilon\} \quad b > 2L$$

$$\mathcal{C}_{b,R,T} = \mathcal{N}_{b\varepsilon} \cap \{|x'| < R\} \cap \{|t| < T\}.$$

Here d is the ordinary distance. We denote by $\Omega_{\varepsilon,R,T}$ a smooth domain with

$$\mathcal{C}_{b/2,R,T} \subset \Omega_{\varepsilon,R,T} \subset \mathcal{C}_{b,R,T}$$

Lemma 3 *(Lemma 3.4 in [FS1]). Let C, c_0, b_0, ω_0 be positive numbers. Choose positive numbers β, γ, δ as above and $\tilde{\alpha}$ such that $0 < \tilde{\alpha} < 1 - \beta$. If $C > 1$ and ω_0 is small enough, there exists a family of functions ϕ_η such that $\phi_\eta \in C^2(\bar{\Omega}_{\varepsilon,R,T})$, $0 \leq \eta \leq 1$ and*

(a) $0 \leq 1 - \omega_0 \leq \phi_\eta \leq 1 + \eta - \omega_0$

(b) $\phi_\eta(\mathcal{L}\phi_\eta - D_t\phi_\eta - |\nabla\phi_\eta|) \geq C(|\nabla\phi_\eta|^2 + \omega_0^2)$

(c) $|D_t\phi_\eta| \leq c\varepsilon^{-\tilde{\alpha}}, |\nabla\phi_\eta| \leq c\varepsilon^{\beta-1}$

(d) $D_t\phi_\eta \geq 0$

(e) $\phi_\eta \leq 1$ *in*

$$\bar{\Omega}_{\varepsilon,R,T} \cap \left(\{-T < t < -T + \varepsilon^{\tilde{\alpha}}\} \cup \{x : R - \frac{\varepsilon^{\tilde{\alpha}/4}}{2} < |x'| < R\}\right)$$

(f) $\phi_\eta \geq 1 - \omega_0 + \eta(1 - c\varepsilon^\beta)$ *in*

$$\bar{\Omega}_{\varepsilon,R,T} \cap \left(\{t > -T + 2\varepsilon^{\tilde{\alpha}}\} \cup \{x : |x'| < R - \frac{\varepsilon^{\tilde{\alpha}/4}}{2}\} \right).$$

Remarks: (1) We will apply this Lemma with ω_0 being the oscillation of the coefficient matrix A_{ij}. Since we are assuming Holder continuous coefficients, we can assume that ω_0 is small, as a rescaling depresses the oscillation.
(2) In [FS1] the ϕ_η are defined as

$$\phi_\eta(x,t) = 1 + \omega_0(|x'|^2 - 1) + \eta \left(\frac{F(x,t) - 1}{2^{\frac{1}{2C-1}} - 1} \right)$$

One clearly sees that the ϕ_η vary continuously in η. In turn, this means that v_η's (defined below) and their free boundaries vary continuously in η as well.
(3) We will be interested in constructing a family of sup-convolutions

$$v_\eta(p) = \sup_{B_{\sigma\phi_\eta(p)}(p)} u_1$$

where $B_{\sigma\phi_\eta(p))}(p)$ is the ball of radius $\sigma\phi_\eta(p)$ centered at p, with σ to be chosen later and $u_1(q) = u(q - \lambda\varepsilon\tau)$. With this in mind, we clarify the role that the conclusions of the lemma have on this family. Condition (b) is the most important. In fact, it is proved in [FS1] that if a function satisfies condition (b), then the function v_η will be a $\mathcal{L}$-subsolution in its positive and negative set. Conditions (e) and (f) involve how the family should behave near the boundary of their domain. Near the boundary no gain in monotonicity can be expected, and hence $\phi_\eta \leq 1$ on this region. In the interior, where gain is expected, we have $\phi_\eta \geq 1 - \omega_0 + \eta(1 - c\varepsilon^\beta)$, and thus there is a definite increase in the radius of the balls over which the supremum is taken.

2.3 Asymptotic Developments

This section describes the behavior of the solution u near its free boundary. The results of this section also find application to the behavior of the sup-convolutions near their zero sets. We explicitly remark that such results are valid for any $\mathcal{L}$-caloric functions vanishing on a distinguished piece of the boundary of a Lipschitz domain. The following result from [FS1] (Lemma 3.5; see also 13.19 in [CS]) provides the first result along these lines.

Lemma 4 *Let u be $\mathcal{L}$-caloric in the open set D, vanishing on $F = \partial D \cap C_1$. Supposed that $(0,0) \in F$ and there is an $(n+1)$-dimensional ball B such that $\bar{B} \cap F = \{(0,0)\}$. Assume that the tangent plane to B is given by*

$$\beta^+ + \alpha^+ \langle x, \nu \rangle = 0$$

for some spatial unit vector ν and some real numbers $\alpha^+, \beta^+, \alpha^+ > 0$, ($-\beta^+/\alpha^+$ finite). Then, either u grows more than any linear function or:

(a) ($B \subset D$) Then, near $(0,0)$, for $t \leq 0$

$$u(x,t) \geq [\beta^+ t + \alpha^+ \langle x, \nu \rangle]^+ + o(d(x,t)).$$

(b) ($B \subset D^C$) Then, near $(0,0)$, for $t \leq 0$

$$u(x,t) \leq [\beta^+ t + \alpha^+ \langle x, \nu \rangle]^+ + o(d(x,t)).$$

Furthermore, equality holds in both case along paraboloids of the form $t = -\gamma \langle x, \nu \rangle$ $\gamma > 0$.

Remark: By applying Lemma 4 to both u^+ and $(-u)^+$, where u is our two-phase viscosity solution, we can infer that if the origin is a right regular point for u^+, and hence a left regular point for $(-u)^+$, then

$$u(x,t) \geq (\beta^+ t + \alpha^+ (x, \nu))^+ - (\beta^- t + \alpha^- (x, \nu))^- + o(d(x,t)).$$

Likewise, if the origin is regular from the left for u we have the same statement with the inequality reversed.

Lemma 5 *Let u be a viscosity solution to our free boundary problem in Q_1 with FB(u) Lipschitz, $(0,0) \in FB(u)$ and suppose that in a neighborhood of the origin with $t \leq 0$ we have for $\alpha^+ > 0$, $\alpha^- \geq 0$*

$$u(x,t) \geq (\beta^+ t + \alpha^+ \langle x, \nu \rangle)^+ - (\beta^- t + \alpha^- \langle x, \nu \rangle)^- + o(d(x,t)).$$

Then $G(\alpha^+, \alpha^-) \leq 1$.

Likewise, if for $\alpha^+ \geq 0$, $\alpha^- > 0$

$$u(x,t) \leq (\beta^+ t + \alpha^+ \langle x, \nu \rangle)^+ - (\beta^- t + \alpha^- \langle x, \nu \rangle)^- + o(d(x,t)).$$

Then $G(\alpha^+, \alpha^-) \geq 1$.

Proof We prove the first statement. Suppose that the conclusion of the lemma is false. Then there exists an $\eta > 0$ such that $G(\alpha^+, \alpha^-) \geq 1 + \eta > 1$. We can assume that $\nu = e_n$ for simplicity.

Let R be a small parabolic neighborhood of the origin. Set

$$\psi(x,t) = \bar{\alpha}^+ x_n + \beta^+ t - ct^2 + \frac{2\Lambda}{\lambda} x_n^2 - \frac{|x'|^2}{2(n-1)}$$

where $\bar{\alpha}^+ = \alpha - \varepsilon$ with ε to be determined later.

Choose $c > 0$ large so that the level surface $\{\psi = 0\}$ is strictly convex and $\{\psi > 0\} \cap R \subset \Omega^+(u)$

Let $\mathcal{L}_r = \frac{1}{r} \sum_{i,j} a_{ij}(rx, rt)$. Observe that

$$\begin{aligned}
\mathcal{L}_r \psi - \psi_t &= \frac{1}{r} \sum_{i,j} a_{ij} D_{ij} \psi - [\beta^+ - 2ct] \\
&= \frac{1}{r} \left[-\frac{1}{n-1}(a_{11} + a_{22} + \ldots + a_{n-1,n-1}) + \frac{2\Lambda}{\lambda} a_{nn} \right] - [\beta^+ - 2ct] \\
&\geq \frac{\Lambda}{r} - [\beta^+ - 2ct] \\
&> 0
\end{aligned}$$

provided r is small enough.

Claim: *If ε and R are small enough, then the function*

$$\phi = \psi^+ - \frac{\bar{\alpha}^-}{\bar{\alpha}^+}\psi^-,$$

with $\bar{\alpha}^- = \alpha^- + \varepsilon$, is a classical strict $\mathcal{L}_r$-subsolution in R.

To prove the claim, note that by the observation above, if r is small enough, ϕ will satisfy the subsolution condition away from its free boundary. So we only need to establish the free boundary condition G.

If ε is sufficiently small, the continuity of G and our assumption that $G(\alpha^+, \alpha^-) \geq 1 + \eta$ implies that

$$G(\phi_n^+, \phi_n^-) \geq G(\alpha^+, \alpha^-) - \eta/2 \geq 1 + \eta/2.$$

Using the continuity of ϕ and G we may assume that the subsolution condition holds throughout R, assuming this region is small enough. That is, along the free boundary of ϕ in R we have

$$G(\phi_\nu^+, \phi_\nu^-)) \geq 1 + \eta/4 > 1.$$

This means that ϕ is a classical strict subsolution in R. Note that this choice of R does not depend on r.

Now define

$$u_r(x,t) = \frac{u(rx, rt)}{r}.$$

Then u_r is a viscosity $\mathcal{L}_r$-solution. Furthermore,

$$u_r(x,t) \geq (\beta^+ t + \alpha^+\langle x, \nu\rangle)^+ - (\beta^- t + \alpha^-\langle x, \nu\rangle)^- + o_r(1)$$

where $o_r(1)$ denotes the decay of the error term as a function of r. If we choose r to be very small, we will then have

$$u_r(x,t) \geq (\beta^+ t + \alpha^+\langle x, \nu\rangle)^+ - (\beta^- t + \alpha^-\langle x, \nu\rangle)^- + o_r(1) > \phi(x,t)$$

on the boundary of our neighborhood R. This strict inequality ought to propagate to the interior since u_r is a viscosity solution and ϕ a subsolution, but we have $u_r(0,0) = 0 = \phi(0,0)$ a contradiction. ∎

Lemma 6 *Let u be a viscosity solution to our free boundary problem with a Lipschitz FB. Define $u_1 = u(p - \lambda\varepsilon\tau)$*

$$v_\eta(x,t) = \sup_{B_{\sigma\phi_\eta}} u_1$$

where ϕ_η is as Lemma 3. Assume that this sup is attained uniformly away from the top and bottom of the ball. Then the following hold:

1. *v_η is a subsolution to our equation on $\Omega^+(v_\eta)$ and $\Omega^-(v_\eta)$.*

2. *All points on $FB(v_\eta)$ are regular from the right.*

3. *$FB(v_\eta)$ is uniformly Lipschitz in space and time*

4. *If $(x_0,t_0) \in FB(v_\eta)$ and $(y_0,s_0) \in FB(u_1)$ with*

$$(y_0,s_0) \in \partial B_{\sigma\phi_\eta(x_0,t_0)}(x_0,t_0)$$

then (x_0,t_0) is a regular point from the right. Moreover if near (y_0,s_0) along the paraboloid $s = s_0 - \gamma\langle y-y_0,\nu\rangle^2$, $(\gamma > 0)$, u_1 has the asymptotic expansion

$$u_1(y,s) = \alpha^+\langle y-y_0,\nu\rangle^+ - \alpha_-\langle y-y_0,\nu\rangle^+ + o(|y-y_0|)$$

$\nu = \frac{y_0-x_0}{|y_0-x_0|}$, then near (x_0,t_0) along the paraboloid $t = t_0 - \gamma\langle x-x_0,\nu\rangle^2$ we have

$$\begin{aligned} v_\eta \geq & \alpha_+(x-x_0,\nu + \frac{\sigma\phi_\eta(x_0,t_0)}{|y_0-x_0|}\nabla(\sigma\phi_\eta))^+ \\ & - \alpha_-(x-x_0,\nu + \frac{\sigma\phi_\eta(x_0,t_0)}{|y_0-x_0|}\nabla(\sigma\phi_\eta))^- + o(|x-x_0|). \end{aligned}$$

Proof (1) is proved in [FS1], Lemma 3.1.

(2) & (3) are standard facts. See Lemma 9.13 in [CS].

(3) For x near x_0 take t to be on the corresponding paraboloid. Set $y = x + \nu\bar{\phi}(x)$ (we suppress the η subscript for convenience; the result is to hold for any choice of η) where

$$\bar{\phi}(x) = \sqrt{\phi^2(x,t) - (s_0-t_0)^2} \leq \phi(x,t).$$

Note that t depends on x here since (x,t) is to lie on the paraboloid. Given this y, let s be the corresponding time value so that (y,s) lies on the paraboloid for u_1. Then $v_\eta(x,t) \geq u_1(y,s)$ since (y,s) is in the ball over which we are taking the sup.

Since

$$\nabla\bar{\phi} \mid_{x_0} = \frac{\phi(x_0,t_0)}{\bar{\phi}(x_0)} \nabla\phi(x_0,t_0).$$

and $\bar{\phi}(x_0) = |x_0 - y_0|$, we can write $\bar{\phi}$ as

$$\bar{\phi}(x) = \bar{\phi}(x_0) + \langle x - x_0, \frac{\phi(x_0,t_0)}{|y_0 - x_0|} \nabla\phi(x_0,t_0)\rangle + o(|x - x_0|).$$

Now we compute

$$\begin{aligned}
\langle y - y_0, \nu\rangle =& \langle x - x_0 + (\bar{\phi}(x) - \bar{\phi}(x_0))\nu, \nu\rangle + o(|x - x_0|) \\
=& \langle x - x_0 + (\langle x - x_0, \frac{\phi(x_0,t_0)}{|y_0 - x_0|} \nabla\phi(x_0,t_0)\rangle)\nu, \nu\rangle + o(|x - x_0|) \\
=& \langle x - x_0, \nu\rangle + (\langle x - x_0, \frac{\phi(x_0,t_0)}{|y_0 - x_0|} \nabla\phi(x_0,t_0)\rangle)\langle \nu, \nu\rangle + o(|x - x_0|) \\
=& \langle x - x_0, \nu + \frac{\phi(x_0,t_0)}{|y_0 - x_0|} \nabla\phi(x_0,t_0)\rangle + o(|x - x_0|).
\end{aligned}$$

Now substitute this result into the asymptotic behavior of u_1 and use the fact that $v_\eta(x,t_0) \geq u_1(y,s_0)$ to reach the desired conclusion.

$\blacksquare$

Remark The condition on the sup being attained uniformly away from the top and bottom of the ball is implicitly used to ensure that $|x_0 - y_0|$ is bounded away from zero in the above computations. The condition is not restrictive; for a solution with Lipschitz free boundary satisfies it near the FB, and by rescaling, one may assume that it occurs throughout C_1.

2.4 Monotonicity up to the Free Boundary

Thus far in this work we have used ε_0-monotone in a direction τ to mean that for any $\varepsilon \geq \varepsilon_0$

$$u(p) - u(p - \varepsilon\tau) \geq c\varepsilon^{\bar{\beta}} u(p).$$

At this point however, it is more convenient to work with a formulation of ε_0-monotonicity more compatible with our sup-convolutions v_η. In this formulation, we say that u is ε_0-monotone in a cone of directions $\Gamma(\theta, e_n)$ if

$$\sup_{B_{\varepsilon \sin \delta}(p)} u(q - \varepsilon\tau) \leq u(p)$$

for any $\tau \in \Gamma(\theta - \delta, e_n)$ and $\varepsilon \geq \varepsilon_0$. Here $\delta = \pi/2 - \theta$ is the defect angle of the cone; if it is zero the cone would be a half-space. Throughout all our work we assume that $\delta \ll \theta$. The two formulations of ε-monotonicity are essentially equivalent, with perhaps a slight difference in ε, so our previous results translated into this formulation become: For $\tau \in \Gamma(\theta - \delta, e_n)$ and $\varepsilon \geq \varepsilon_0$

$$\sup_{B_{\varepsilon \sin \delta}(p)} u(q - \varepsilon\tau) \leq (1 - c\varepsilon^{\bar{\beta}})u(p).$$

Our goal in what follows is to decrease ε by a factor of $\lambda < 1$. Let $\sigma = \varepsilon(\sin\delta - (1-\lambda))$. Now if λ is close enough to one

$$B_\sigma(p - \lambda\varepsilon\tau) \subset B_{\varepsilon \sin \delta}(p - \varepsilon\tau).$$

This means

$$\sup_{B_\sigma(p)} u(q - \lambda\varepsilon\tau) \leq (1 - c\varepsilon^{\bar{\beta}})u(p).$$

This has decreased ε but at the expense of reducing the ball's radius.

However, away from the free boundary at a distance of $(M\varepsilon)^\gamma$, by Lemma 2 our solution u is in fact fully monotone. So away from the free boundary there is no need to give up any of the radius and we have by the definition of monotone

$$\sup_{B_{\varepsilon\lambda \sin \delta}(p)} u(q - \lambda\varepsilon\tau) \leq u(p).$$

We will use the variable family of radii v_η to bridge these two conclusions. Recall $\tilde{\alpha}$ occurs in the definition of the ϕ_η and $\bar{\beta} = 1 - \gamma + \beta$ is defined in Section 2.

Theorem 2 *Let u be a solution to our FBP in $C_{R,T} = B'_R \times (-T, T)$ such that* *(i) u is monotonically increasing along the directions of a spacial cone $\Gamma^x(\theta, e_n)$ with $\pi/2 - \theta \ll 1$.*

(ii) u is ε-monotone in a space time cone of directions $\Gamma(\theta_*, e_n)$

(iii) u is monotone along the directions $\tau \in \Gamma(\theta_*, e_n)$ outside an ε-neighborhood of $FB(u)$.

(iv) The non-degeneracy condition holds for u at regular points from the right.

Then there exists an ε_0 and a λ, $0 < \lambda < 1$ such that if $\varepsilon \leq \varepsilon_0$ we have in $C_{R-c\varepsilon^{\bar{\alpha}}, T-c\varepsilon^{\bar{\alpha}}}$ u is $\lambda\varepsilon$-monotone in $\Gamma(\theta_* - \bar{c}\varepsilon^{\beta}, e_n)$.

Proof For the purposes of this proof, let $u_1(p) = u(p - \lambda\varepsilon\tau)$.

Since we have (*iii*), we only need to show the improvement in an ε-neighborhood of the free boundary. Precisely, our goal is to show the improvement in $\Omega_{\varepsilon,R,T} \cap C_{R-c\varepsilon^{\bar{\alpha}}, T-c\varepsilon^{\bar{\alpha}}}$.

Define

$$v_\eta(p) = \sup_{B_{\sigma\phi_\eta}(p)} u(q - \lambda\varepsilon\tau).$$

Choose $\bar{\eta}$ such that

$$\sigma(1 - \omega_0 + \bar{\eta}) = \varepsilon(\lambda \sin\delta - c\varepsilon^{\beta})$$

where $\sigma = \varepsilon(\sin\delta - (1-\lambda))$. Assume our defect angle δ is small and take $(1-\lambda) = \frac{1}{2}\sin\delta$. Then, since $0 < \varepsilon \ll \delta \ll 1$, we have that $1/3 < \bar{\eta} < 1$ (the lower bound is the one of consequence here). We proceed to perturb this function as follows:

$$\bar{v}_\eta = v_\eta + c\varepsilon^{1+\beta-\gamma} w_\eta.$$

We define w_η as follows:

$$\begin{cases} \mathcal{L}w_\eta - (w_\eta)_t = 0 & \text{in} \quad \Omega^+(v_\eta) \cap \Omega_{\varepsilon,R,T} \\ w_\eta = 0 & \text{on} \quad FB(v_\eta) \\ w_\eta = u & \text{on rest of} \quad \partial_p[\Omega_{\varepsilon,R,T} \cap \Omega^+(v_\eta)]. \end{cases}$$

Extend w_η by zero on the rest of $\Omega_{\varepsilon,R,T}$. Note that $w_\eta \leq u$ in $\Omega^+(v_\eta) \cap \Omega_{\varepsilon,R,T}$ by the maximum principle.

We will show that for every $\eta \in [0, \bar{\eta}]$ we have

$$\bar{v}_\eta \leq u$$

in $\Omega_{\varepsilon,R,T} \cap C_{R-c\varepsilon^{\bar{\alpha}}, T-c\varepsilon^{\bar{\alpha}}}$. This will be accomplished by showing that the set of η's for which $\bar{v}_\eta \leq u$ is non-empty and both open and closed. The set being closed follows from the fact that the $\bar{v_\eta}$ vary continuously in η (see the remark after Lemma 3), so we only need to show that it is non-empty and open.

Non-Empty: We show that $\bar{v}_0 \leq u$. Now from the properties of ϕ_η, $\phi_0 \equiv 1 - \omega_0$, so that

$$v_0 = \sup_{B_{\sigma(1-\omega_0)}} u(q - \lambda\varepsilon\tau).$$

Now clearly $\sigma \geq (1-\omega_0)\sigma$. Using the fact that

$$\sup_{B_\sigma(p)} u(q - \lambda\varepsilon\tau) \leq (1 - c\varepsilon^{\bar{\beta}})u(p),$$

we see that $v_0(p) \leq (1 - c\varepsilon^{1+\beta-\gamma})u(p)$ (recall that $1 + \beta - \gamma = \bar{\beta}$). Now by choosing a suitable new constant c, we can arrange $\bar{v}_0 = v_0 + c\varepsilon^{1+\beta-\gamma} w_0 \leq u$.

Open: We claim that it will be enough to show that

$$\Omega_{\varepsilon,R,T} \cap \{\bar{v}_\eta > 0\} \subset\subset \Omega_{\varepsilon,R,T} \cap \{u > 0\} \tag{2.1}$$

for every $\eta \in [0, \bar{\eta})$. We argue this claim as follows:

First, note that u is fully monotone, and thus $\lambda\varepsilon$-monotone, outside of $\Omega_{\varepsilon,R,T}$ by hypothesis (iii). By our choices of σ and $\bar{\eta}$ at the beginning of this proof, we have that $\sigma\phi_\eta \leq \lambda\varepsilon \sin\delta$ for any $0 \leq \eta \leq 1$. This means that outside of $\Omega_{\varepsilon,R,T}$

$$v_\eta(p) \leq \sup_{B_{\lambda\varepsilon\sin\delta}} u(p - \lambda\varepsilon\tau) \leq (1 - c(\lambda\varepsilon)^{\bar{\beta}})u(p)$$

for any $\eta \in [0,1]$. Using this, the assumption that λ is close to one, and adjusting the constant c that appears in the definition of $\bar{v}_\eta$, we have that $\bar{v}_\eta \le u$ along the boundary of $\Omega_{\varepsilon,R,T}$.

Now assume $\Omega_{\varepsilon,R,T} \cap \{\bar{v}_\eta > 0\} \subset\subset \Omega_{\varepsilon,R,T} \cap \{u > 0\}$. We want to show that $\bar{v}_\eta \le u$. By the preceding argument, we have that $u \ge \bar{v}_\eta$ on $\partial_p[\{\bar{v}_\eta > 0\} \cap \Omega_{\varepsilon,R,T}]$. Inside this domain both functions are classical solutions so the maximum principle implies that $u \ge \bar{v}_\eta$ inside as well. An entirely similar argument, though somewhat simpler since w_η does not appear, yields the same conclusions for $\{\bar{v}_\eta < 0\}$. Hence $\bar{v}_\eta \le u$ everywhere.

We now proceed to prove (2.1) by contradiction. Assume there exists a point $(x_0, t_0) \in FB(u) \cap FB(v_\eta)$ for some η which we now regard as fixed. All points on $FB(v_\eta)$ are regular from the right and, since the $FB(u)$ touches $FB(v_\eta)$ at (x_0, t_0), we have that this point is regular from the right for u.

By the definition of v_η, we have that there is a corresponding point $(y_0, s_0) \in \partial B_{\sigma\phi_\eta(x_0,t_0)}(x_0, t_0) \cap FB(u_1)$. This point is then a regular point from the left for u_1. By Lemma 4 and Lemma 5 we have along parabolas of the form $s = s_0 - \gamma\langle y - y_0, \nu\rangle^2$

$$u_1(y, s) = \alpha_+\langle y - y_0, \nu\rangle^+ - \alpha_-\langle x, \nu\rangle^- + o(d(x, t))$$

where $\nu = \frac{y_0 - x_0}{|y_0 - x_0|}$ with $G(\alpha_+, \alpha_-) \ge 1$.

Next, by Lemma 6, we have that near (x_0, t_0) along the the parabola $t = t_0 - \gamma\langle x - x_0, \nu\rangle^2$ we have

$$\begin{aligned} v_\eta \ge & \alpha_+(x - x_0, \nu + \frac{\sigma\phi_\eta(x_0, t_0)}{|y_0 - x_0|}\nabla(\sigma\phi_\eta))^+ \\ & - \alpha_-(x - x_0, \nu + \frac{\sigma\phi_\eta(x_0, t_0)}{|y_0 - x_0|}\nabla(\sigma\phi_\eta))^- + o(|x - x_0|). \end{aligned}$$

In other words,

$$v_\eta \ge \alpha_+|\nu^*|(x - x_0, \nu^*)^+ - \alpha_-|\nu^*|(x - x_0, \nu^*)^+ + o(|x - x_0|),$$

where $\nu^* = \nu + \frac{\sigma\phi_\eta(x_0,t_0)}{|y_0 - x_0|}\nabla(\sigma\phi_\eta)$.

We now turn to the behavior of w_η and invoke the non-degeneracy condition we have on u^+. This implies a non-degeneracy condition on u_1, and in turn on v_η via Lemma 6 . Precisely, by non-degeneracy, the α_+ appearing in the asymptotic behavior of u_1 is strictly positive bounded away from zero (a consequence of Lemma 4). Since the same α_+ appears in the asymptotic behavior of v_η, v_η also has this non-degeneracy. By using our Boundary Harnack Principle for $\mathcal{L}$-caloric functions on Lipschitz domains with w_η and v_η we deduce that w_η also possesses a non-degeneracy property, $(w_\eta)_{\nu^*} \geq c > 0$ along $FB(v_\eta) \cap C_{R-c\varepsilon^{\tilde{\alpha}},T-c\varepsilon^{\tilde{\alpha}}}$.

Now combining the behavior of v_η and w_η we have

$$\bar{v}_\eta \geq \bar{\alpha}_+(x-x_0,\nu)^+ - \alpha_-(x-x_0,\nu)^- + o(|x-x_0|)$$

with

$$\begin{aligned}\bar{\alpha}_+ &\geq \alpha_+(1-c\sigma^2|\nabla\phi_\eta|) + c\varepsilon^{1+\beta-\gamma}\\ &\geq \alpha_+(1-c\varepsilon^2\delta\varepsilon^{\beta-1}) + c\varepsilon^{1+\beta-\gamma}\\ &> \alpha_+\end{aligned}$$

for ε small enough. We have only used w to perturb the positive part of v_η, hence the α_- is unchanged.

Since, as noted above, (x_0,t_0) is a regular point from the right for u, we have by Lemma 4

$$u(x,t_0) \geq \alpha_+^{(2)}(x-x_0,\nu)^+ - \alpha_-^{(2)}(x-x_0,\nu)^- + o(|x-x_0|)$$

where $G(\alpha_+^{(2)},\alpha_-^{(2)}) \leq 1$.

G is strictly increasing in the first argument so

$$G(\alpha_+^{(2)},\alpha_-^{(2)}) \leq 1 \leq G(\alpha_+,\alpha_-) < G(\bar{\alpha}_+,\alpha_-)$$

However, $u-\bar{v}_\eta$ is a nonnegative supersolution in $\Omega^+(\bar{v}_\eta)$, and so we must have $\alpha_-^{(2)} \leq \alpha_-$. Additionally, by the Hopf Principle, $\alpha_+^{(2)} > \bar{\alpha}_+$. But G is strictly increasing in its first argument, strictly decreasing in its second and thus we arrive at a contradiction. Hence the set is open.

Conclusions: We have proved that $\bar{v}_\eta \leq u$ for any $\eta \in [0, \bar{\eta}]$. In particular, $\bar{v}_{\bar{\eta}} \leq u$.

Now recall from the construction of the family ϕ_η that

$$\phi_\eta \geq 1 - \omega_0 + \eta(1 - c\varepsilon^\beta)$$

in

$$\bar{\Omega}_{\varepsilon,R,T} \cap \left(\{t > -T = \varepsilon^{\tilde{\alpha}}\} \cup \{x : |x'| < R - \frac{\varepsilon^{\tilde{\alpha}/4}}{2}\} \right).$$

Since

$$\sigma(1 - \omega_0 + \bar{\eta}) = \varepsilon(\lambda \sin\delta - c\varepsilon^\beta)$$

and

$$\sigma = \frac{\varepsilon}{2} \sin\delta$$

we have $\sigma\phi_{\bar{\eta}} \geq \varepsilon(\lambda \sin\delta - c^*\varepsilon^\beta)$ in this region (here we used the bound$\bar{\eta} > 1/3$).

Finally, $\lambda \sin\delta - c^*\varepsilon^\beta > \lambda \sin(\delta - \bar{c}\varepsilon^\beta)$. We conclude that on

$$\bar{\Omega}_{\varepsilon,R,T} \cap \left(\{t > -T = \varepsilon^{\tilde{\alpha}}\} \cup \{x : |x'| < R - \frac{\varepsilon^{\tilde{\alpha}/4}}{2}\} \right)$$

u is $\lambda\varepsilon$-monotone for any direction $\tau \in \Gamma(\theta - \bar{c}\varepsilon_0^\beta, e_n)$.

■

Remark: Hypothesis (i) is not used in the proof of the theorem, but it is needed in order to apply Lemma 2 in the proof of the following corollary.

Corollary 1 *Let u be as in Theorem 2 on $C_{1,1}$. Then on a smaller cylinder $C_{5/6,5/6}$ u is fully monotone in a space-time cone of directions. As a result, we have that*

$$|u_t| \leq C D_n u$$

in this region.

Proof From Theorem 2 we can conclude that u is $\lambda\varepsilon$-monotone in the directions $\Gamma(\theta - \bar{c}\varepsilon^\beta, e_n)$. From Lemma 2, $(M\lambda\varepsilon_0)^\gamma$ away from the free boundary u is fully monotone in the directions $\tau' = \tau + c(M\lambda\varepsilon_0)^{\alpha+\delta-1} e_n$. So we have that u is fully monotone $(M\lambda\varepsilon)^\gamma$ away from the free boundary in the cone of directions $\Gamma(\theta - \bar{c}\varepsilon^\beta - (M\varepsilon)^{(\alpha+\delta-1)/2}, e_n)$.

We iterate this. We can achieve full monotonicity of u up to the free boundary on the smaller cylinder provided we can control the cone loss at every step. This amounts to

$$\theta^t = \theta^t_* - \bar{c}\varepsilon_0^\beta \sum_k \lambda^{\beta k} - (M\varepsilon_0)^{(\alpha+\delta-1)/2} \sum_k c\lambda^{(\alpha+\delta-1)/2} > 0$$

and

$$\varepsilon_0^{\tilde{\alpha}} \sum \lambda^{k\tilde{\alpha}} < \frac{1}{6}.$$

We observe that this last term controls how far we stay away from the sides of the original cylinder.

All the sums involved are convergent geometric series, so provided that ε_0 is small enough we can achieve both conditions. ■

2.5 Lipschitz Continuity of Solution

The existence of a space-time monotonicity cone up to the free boundary established in Corollary 1 allows us to prove Theorem 1.

Proof [**Proof of Theorem 1**] We first note that Lemmas 5.1 and 5.2 from [FS1] continue to hold in our case.

By Corollary 1

$$|u_t| \leq C|\nabla u|,$$

and thus it suffices to show only the boundedness of the spacial gradient across the free boundary.

Let $(x_0, t_0) \in \Omega^+(u) \cap C_{1/2}$ be a point such that $\text{dist}((x_0,t_0), FB(u)) \leq d_0$. Then the $(n+1)$-dimensional ball $B_d(x_0,t_0)$ touches the free boundary at some point $(\bar{x},\bar{t})$, which we will assume to be the origin for simplicity. We then let $h = \text{dist}((x_0,0),(0,0))$. Since the free boundary is Lipschitz, we have that there exists a c (not dependent on (x_0,t_0)) such that $cd \leq h \leq d$.

Now set $A = \frac{u(x_0,0)}{h}$. We aim to prove that A is bounded independent of the point (x_0, t_0). Having thus set up the parameters of the proof, we proceed as in [FS1], and we refer the reader to that source for more details. Seeking contradiction we construct a function z in a small neighborhood of the origin Q_s which is a subsolution to the equation $\mathcal{L} - \partial_t$. In addition, using that at the origin the spacial normal (by construction) is e_n, one can show that z satisfies the properties

$$z_\nu^+ = z_n^+ \geq cA \quad \text{and} \quad z_\nu^- = z_n^- \leq \frac{c_1}{As^2},$$

and

$$\frac{z_t^+(0,0)}{z_n^+(0,0)} = \beta.$$

The last condition essentially describes the fact that β is the speed of the free boundary for z. We will show that A too large forces a contradiction.

We require that β satisfy

$$\frac{1}{3}\lambda\tilde{C}A < \beta < \lambda\tilde{C}A.$$

Note that A large implies that β is large.

We recall that $G(u_\nu^+, u_\nu^-)$ is increasing in the first argument and decreasing in the second. So from the estimates on z at the origin above we have

$$G(z_n^+, z_n^-) \geq G(cA, \frac{c_1}{As^2})$$

By increasing A we can make the right hand side as large as we desire. In particular, we can make $G(cA, \frac{c_1}{As^2}) > 1$. By the continuity of z we obtain that z is a strict subsolution to our problem in a neighborhood of the origin. We can now adjust the speed β to be fast enough (possibly increasing A further if need be) so that the free boundary of z stays to the right (i.e. in the positivity set of) the free boundary u. This is possible since the speed of u is finite, owing to u having Lipschitz free boundary. By construction $z < u$ on the parabolic boundary of this neighborhood so the same should hold inside by virtue of u being a viscosity solution. But $u(0,0) = z(0,0)$, a contradiction. ∎

3. Regularity of the Free Boundary

3.1 Definitions and Statement of Results

Our starting point in the analysis of the free boundary will be the following result proved in Chapter 2 of this work. We denote the cone of directions with opening θ and axis η by $\Gamma(\theta, \eta)$. We have slightly restated the theorem to emphasize the existence of a cone of monotonicity for the solution, a fact that will be central to the analysis that follows.

Theorem *Let u be a viscosity solution to* (1.1) *satisfying the hypotheses of this section. Then u is Lipschitz and possesses a space-time cone of directions with axis e_n and opening angle θ in which the solution is monotone:*

$$u(x - \tau) \leq u(x) \quad \forall \tau \in \Gamma(\theta, e_n)$$

The main result of this chapter is the following theorem.

Theorem 2 *Let u be a solution to our free boundary problem in Q_1 satisfying the hypotheses of this section. Then for every point (x, t) on the free boundary in $Q_{1/2}$ there exists a normal vector to the surface $\eta(x, t)$. Furthermore, this normal vector satisfies*

1. $|\eta(x, t) - \eta(y, t)| \leq C|x - y|^\alpha$
2. $|\eta(x, s) - \eta(x, t)| \leq C|s - t|^\beta$

Finally, the free boundary condition is taken up with continuity by the solution u so that u is a classical solution to (1.1).

3.2 Preliminaries

3.2.1 Main Tools

We reiterate the main tools used in the analysis for the reader's convenience.

Define the domain Ω_{2r} by

$$\Omega_{2r} = \{(x', x_n, t) : |x'| < 2L^{-1}r, |t| < 4L_0^{-2}r^2, f(x', t) < x_n < 4r\}.$$

Denote by $P_r = (0, r, 0)$, $\overline{P_r} = (0, r, 2L_0^{-2}r^2)$, $\underline{P_r} = (0, r, -2L_0^{-2}r^2)$. These are the inward point, forward point and backward point, respectively. Denote by $\delta(X, Y)$ the parabolic distance between $X = (x, t)$ and $Y = (y, s)$ and by δ_X the parabolic distance from X to the origin.

Our tools, valid for $\mathcal{L}$-caloric functions on Lipschitz domains vanishing on a piece of the boundary, are as follows (see [FS2]):

Interior Harnack Inequality: There exists a positive constant $c = c(n, \lambda, \Lambda)$ such that for any $r \in (0, 1)$

$$u(\underline{P_r}) \leq cu(\overline{P_r}).$$

Carleson Estimate: There exists a $c = c(n, \lambda, \Lambda, L, L_0)$ and $\beta = \beta(n, \lambda, \Lambda, L, L_0)$, $0 < \beta \leq 1$ such that for every $X \in \Omega_{r/2}$

$$u(X) \leq c\left(\frac{\delta_X}{r}\right)^{\beta} u(\overline{P_r}).$$

Boundary Harnack Principle: There exists $c = c(n, \lambda, \Lambda, L, L_0)$ and $\beta = \beta(n, \lambda, \Lambda, L, L_0)$, $0 < \beta \leq 1$, such that for every $(x, t) \in \Omega_{2r}$ and u and v are two solutions

$$\frac{u(x,t)}{v(x,t)} \geq c\frac{u(\underline{P_r})}{v(\overline{P_r})}.$$

Backward Harnack Inequality: Let $m = u(\underline{P_{3/2}})$ and $M = \sup_{\Omega_2} u$. Then there exists a positive constant $c = c(n, \lambda, \Lambda, L, L_0, M/m)$ such that if $r \leq 1/2$

$$u(\overline{P_r}) \leq cu(\underline{P_r}).$$

Throughout the work we will use c to denote constants which depend on some or all of $n, \lambda, \Lambda, L, L_0, M/m$.

3.2.2 Initial Configurations and Domains

In what follows it will be necessary to know that the coefficients $a_{ij}(x,t)$ in the operator $\mathcal{L}$ are suitably close to δ_{ij}. To this end we define $a^s_{ij}(x,t) = a_{ij}(sx, s^2t)$ and set

$$\mathcal{L}^s - \partial_t = \sum_{i,j} a^s_{ij}(x,t) D_{ij} - \partial_t$$

to be the parabolic operator with these dilated coefficients. Set

$$u_s(x,t) = \frac{u(sx, s^2t)}{s}.$$

Then we have the equivalence

$$\mathcal{L}u - u_t = 0 \Leftrightarrow \mathcal{L}^s u_s - (u_s)t = 0.$$

Note that this parabolic rescaling of u does not alter the free boundary condition in (1.1). We will assume $a_{ij}(0,0) = \delta_{ij}$ and set $A = \sup_{i,j}[a_{ij}]_\alpha$.

Let $x_0 = \frac{3}{4}e_n$, $P_0 = (x_0, 0)$, $\overline{P_0} = (x_0, \frac{9}{8L_0^2})$, $\underline{P_0} = (x_0, -\frac{9}{8L_0^2})$. These are inward, forward and backward reference points, respectively.

Next define regions $T = B'_{1/4}(x_0) \times (-\frac{9}{16L_0^2}, \frac{9}{16L_0^2})$ and set

$$\Phi = B'_{1/4-\sigma}(x_0) \times (-\frac{9}{16L_0^2} + \sigma^2, \frac{9}{16L_0^2} - \sigma^2)$$

σ to be specified later. By construction the parabolic distance from $\partial\Phi$ to ∂T is σ.

Finally, let

$$\Psi = B'_{1/8}(x_0) \times (-\frac{9}{32L_0^2}, \frac{9}{32L_0^2}).$$

In what follows we will have $\Psi \Subset \Phi \Subset T$ by our choice of σ. Additionally, by an initial change of variables $u(rx, rt)$, with $r < 1$, we can reduce the Lipschitz constants L and L_0 to be less than one, so that the above regions and test points are contained within Ω_4. Finally, by a rescaling we have the free boundary of u contained in $\{|x_n| < 1/10\}$.

Next define z by

$$\begin{aligned} \Delta z - z_t = 0 \quad & \text{in } T \\ z = u \quad & \text{on } \partial_p T. \end{aligned}$$

Note that

$$\mathcal{L}^s(u-z)-(u-z)_t=\sum(a^s_{ij}-\delta_{ij})D_{ij}z$$
$$\Delta(u-z)-(u-z)_t=\sum(a^s_{ij}-\delta_{ij})D_{ij}u.$$

We may assume by this configuration that the conclusion of Lemma 2.1 in [FS1] holds throughout Ω_4. This states that

$$c_1\frac{u(X)}{d_X}\le D_n u(X)\le c_2\frac{u(X)}{d_X}$$

where here d_X denotes the distance from $X=(x,t)$ to the FB at time level t.

3.3 Interior Enlargement of the Monotonicity Cone

The results in this section only require the following: u is $\mathcal{L}^s$-caloric, where $\mathcal{L}^s$ is suitably close to Δ (as controlled by the a^s_{ij}), u vanishes on the piece of the boundary $\{f(x',t)=x_n\}$, and u is Lipschitz with a monotonicity cone $\Gamma(e_n,\theta)$. In particular, the free boundary condition $G(u^+_\nu,u^-_\nu)=1$ plays no role in these results. Our method of proof is similar to [CFS] in the elliptic case.

Lemma 7 *Let u be a solution to $\mathcal{L}^s-u_t=0$ in Ω_4, z as above. Then*

$$|u-z|^*_{2+\alpha,\Phi}\le Ks^\beta u(\underline{P_0}) \tag{3.1}$$

where $K=K(A)$ is a constant which depends on A as well as the usual quantities and $\beta=\frac{\alpha^2}{\alpha+2}$.

Proof We apply the Schauder estimates to the difference $u-z$ to obtain

$$|u-z|^*_{2+\alpha,\Phi}\le C(|u-z|_{0,\Phi}+|\sum(a_{ij}-\delta_{ij})D_{ij}u|^{(2)}_{0,\alpha,\Phi}) \tag{3.2}$$

using the standard (see [L]) notation for these norms and weighted norms. Recall that $|f|_\alpha^{(2)} = |f|_0^{(2)} + [f]_\alpha^{(2)}$. We begin by estimating the Hölder norm term as follows:

$$\begin{aligned}
&|(a_{ij}^s - \delta_{ij})D_{ij}u|_{0,\Phi}^{(2)} + [(a_{ij}^s - \delta_{ij})D_{ij}u]_{\alpha,\Phi}^{(2)} \\
&\leq As^\alpha |D_{ij}u|_{0,\Phi}^{(2)} + |(a_{ij}^s - \delta_{ij})|_{0,\Phi}[D_{ij}u]_{\alpha,\Phi}^{(2)} + [(a_{ij}^s - \delta_{ij})]_{\alpha,\Phi}|D_{ij}u|_{0,\Phi}^{(2)} \\
&\leq As^\alpha |D_{ij}u|_{0,\Phi}^{(2)} + As^\alpha [D_{ij}u]_{\alpha,\Phi}^{(2)} + As^\alpha |D_{ij}u|_{0,\Phi}^{(2)} \\
&\leq As^\alpha (|u|_{2+\alpha,\Phi}^*) \leq CAs^\alpha |u|_{0,\Phi} \leq CAs^\alpha u(\overline{P_0}) \\
&\leq CAs^\alpha u(\underline{P_0}).
\end{aligned}$$

The Backward Harnack Inequality was used to obtain the last inequality. Now we estimate the sup norm term in (3.2). Using the *a priori* estimates we have

$$|u - z|_{0,\Phi} \leq |u - z|_{0,\partial_p \Phi} + C' \sup_\Phi |(a_{ij}^s - \delta_{ij})D_{ij}u|.$$

The first term is estimated as follows: Recall that $u = z$ on the boundary of T so their difference is zero. Now $\partial_p \Psi$ lies a distance σ from ∂T, so using the Hölder continuity up to the boundary we have that $|u - z|_{0,\partial_p \Psi} \leq c\sigma^\alpha |u - z|_{0,T}$.

Using this we have

$$\begin{aligned}
|u - z|_{0,\Phi} &\leq c\sigma^\alpha |u - z|_{0,T} + C' s^\alpha A \sigma^{-2} |u|_{0,\Phi} \\
&\leq c\sigma^\alpha |u|_{0,T} + C' s^\alpha A \sigma^{-2} |u|_{0,\Phi}.
\end{aligned}$$

Select $\sigma = s^{\frac{\alpha}{\alpha+2}}$ and obtain

$$|u - z|_{0,\Phi} \leq s^{\frac{\alpha^2}{\alpha+2}} |u|_{0,T}(c + C'A) \leq c' s^{\frac{\alpha^2}{\alpha+2}} u(\overline{P_0})(c + C'A).$$

Now we always have $\alpha > \frac{\alpha^2}{\alpha+2}$ for $\alpha > 0$ so for $s < 1$ we have $s^\alpha < s^{\frac{\alpha^2}{\alpha+2}}$. Combining this with the estimate for the Hölder norm above we obtain

$$|u - z|_{2+\alpha,\Phi}^* \leq [CA + c'(c + C'A)]\, s^{\frac{\alpha^2}{\alpha+2}} u(\overline{P_0}).$$

This is the conclusion of the lemma with $K = (CA + c'(c + C'A))$. ■

At this point it becomes convienent to begin treating the spacial portion of the cone and space-time cone seperately. We denote these by $\Gamma_x(e_n, \theta_x)$ and $\Gamma_t(\eta, \theta_t)$

respectively, η a vector in the $e_n - e_t$ plane. We now focus on expanding these cones of directions.

Lemma 8 *Let u be a solution to $\mathcal{L}^s - u_t = 0$ in Ω_4 with a cone of monotonicity $\Gamma(e_n, \theta)$. Let $\nabla = \frac{1}{|\nabla u(\underline{P_0})|}\nabla u(\underline{P_0})$. Then if s is sufficiently small, for any $\tau \in \Gamma_x(e_n, \theta)$, $|\tau| = 1$,*

$$D_\tau u(X) \geq (C\langle \nabla, \tau\rangle - cs^\beta)u(\underline{P_0}) \tag{3.3}$$

for all $X \in \Psi$. The same statement holds for $\tau \in \Gamma_t(\eta, \theta_t)$ with ∇ the unit vector in direction $(u_{x_n}(\underline{P_0}), u_t(\underline{P_0}))$ in the $e_n - e_t$ plane.

Proof The proof follows the same lines in both the spacial and space-time cases. We begin with the spacial case.

Let z be as in the previous lemma so that

$$|u - z|^*_{2+\alpha,\Phi} \leq cs^\beta u(\underline{P_0})$$

where $\beta = \frac{\alpha^2}{\alpha+2}$.

Then $D_\tau z + cs^\beta u(\underline{P_0})$ is a non-negative solution to the heat equation in Φ. By the Harnack Inequality, for (x,t) in Ψ we have

$$D_\tau z(x,t) + cs^\beta u(\underline{P_0}) \geq c'\left(D_\tau z(\underline{P_0}) + cs^\beta u(\underline{P_0})\right). \tag{3.4}$$

Hence, letting $\nabla' = \frac{1}{|\nabla z(\underline{P_0})|}\nabla z(\underline{P_0})$, and assuming without loss of generality that $c' < 1$, we obtain

$$\begin{aligned} D_\tau z &\geq c' D_\tau z(\underline{P_0}) - cs^\beta u(\underline{P_0}) & (3.5)\\ &= c'|\nabla z(\underline{P_0})|\langle \nabla', \tau\rangle - cs^\beta u(\underline{P_0}). & (3.6)\end{aligned}$$

Now using the Schauder estimate, $\frac{u}{d} \sim |\nabla u|$ and the Harnack inequality we have

$$|\nabla z| \geq |\nabla u| - cs^\beta u(\underline{P_0}) \geq (C - cs^\beta)u(\underline{P_0}).$$

So if s is small enough then $|\nabla z(\underline{P_0})| \geq cu(\underline{P_0})$ (4.6) becomes

$$D_\tau z \geq (c^*\langle \nabla', \tau\rangle - cs^\beta)u(\underline{P_0}). \tag{3.7}$$

Using the Schauder estimate once more we have

$$|\nabla' - \nabla| \leq \frac{|\nabla u(\underline{P_0}) - \nabla z(\underline{P_0})|}{|\nabla u(\underline{P_0})|} + \frac{|(|\nabla z(\underline{P_0})| - |\nabla u(\underline{P_0})|)|}{|\nabla u(\underline{P_0})|} \leq cs^{\beta}.$$

This is proved as follows: After adding and subtracting the quantity $|\nabla z|\nabla z$, the left-hand side becomes (suppressing the dependence on the point)

$$\frac{|\nabla z|\nabla u| - \nabla u|\nabla z||}{|\nabla z||\nabla u|} = \frac{||\nabla z|(\nabla z - \nabla u) + \nabla z(|\nabla u| - |\nabla z|)|}{|\nabla z||\nabla u|}.$$

Applying the triangle inequality and canceling terms we obtain the first inequality. The second one is then a consequence of the Schauder estimate (with a different constant c).

We have thus established $\langle \nabla', \tau \rangle \geq \langle \nabla, \tau \rangle - cs^{\beta}$. Replacing this in (3.7) we have

$$D_{\tau} z \geq (c_1 \langle \nabla, \tau \rangle - cs^{\beta}) u(\underline{P_0}).$$

Finally, using the Schauder estimate one last time we obtain (with different constants than in the previous line)

$$D_{\tau} u \geq (C \langle \nabla, \tau \rangle - cs^{\beta}) u(\underline{P_0}).$$

In the space-time case the same calculation works with ∇ the unit vector in $e_n - e_t$ plane in direction $(u_{x_n}(\underline{P_0}), u_t(\underline{P_0}))$ and ∇' the vector in direction $(z_{x_n}(\underline{P_0}), z_t(\underline{P_0}))$.
■

Remark: In the case of the heat equation the inequality (3.3), without the 'error' term $cs^{\beta}u(\underline{P_0})$, can be obtained easily by simply applying the Harnack Inequalities to the solution. In the variable coefficient case (3.3) acts as a substitute.

At this point we need to make sure that $D_{\tau} u$ remains positive, which cannot be guaranteed because of the error term in (3.3). To deal with this we eliminate the portion of the original cone consisting of the vectors which make an angle of more than $\frac{99\pi}{200}$ with ∇. We denote this modified set of directions with $\Gamma'_x(e_n, \theta_x)$ (or $\Gamma'_t(\eta, \theta_t)$ as the case may be). Then for some c_3 and any $\tau \in \Gamma'_x(e_n, \theta_x)$

$$\langle \nabla, \tau \rangle \geq c_3 \delta,$$

where $\delta = \frac{\pi}{2} - \theta_x$ is the defect angle of the cone, c_3 depends on how much of the cone was deleted. In the space-time case we use $\mu = \frac{\pi}{2} - \theta_t$ to denote the defect angle; initially this is the same as δ but this will not hold in the iteration later in the paper.

As it is by now standard, this monotonicity can be described in terms of the sup-convolution, in our case over 'thin' balls either purely spacial or in the space-time plane. Precisely,

$$v_\varepsilon(X) = \sup_{B'_\varepsilon(X)} u(Y - \tau) \leq u(X)$$

for any $\tau \in \Gamma'(e_n, \frac{\theta}{2})$ sufficiently small, with $\varepsilon = |\tau| \sin \frac{\theta}{2}$. The B' denotes a thin ball either purely in space or in space-time, depending on whether τ is in Γ'_x or Γ'_t.

In what follows, the direction τ is either in Γ'_x or Γ'_t; the proofs are the same. We distinguish between them only later work will make a distinction and it is convenient to have interior enlargement respect this distinction.

Lemma 9 *Let u be as in Lemma 8. Then there exists $s_0 > 0$ such that if $s \leq s_0$ we have*

$$u(\underline{P_0}) - v_\varepsilon(\underline{P_0}) \geq \sigma \varepsilon u(\underline{P_0}). \tag{3.8}$$

Proof If $Y \in B_\varepsilon(\underline{P_0})$ then, invoking the Mean Value Theorem with $\bar{\tau} = \tau + (\underline{P_0} - Y)$, we obtain

$$u(Y - \tau) = u(\underline{P_0} - \bar{\tau}) = u(\underline{P_0}) - |\bar{\tau}| D_{\bar{\tau}} u(X^*). \tag{3.9}$$

We estimate $D_{\bar{\tau}} u$ from below. If $\tau \in \Gamma(e_n, \frac{\theta}{2})$ then $\bar{\tau} \in \Gamma'(e_n, \theta)$, so using the observation immediately preceding this lemma we have

$$\begin{aligned} D_{\bar{\tau}} u &\geq (c_1 \langle \nabla, \bar{\tau} \rangle - c_2 s^\beta) u(\underline{P_0}) \\ &\geq c \delta u(\underline{P_0}) \end{aligned}$$

for any $s \leq s_0 = \left(\frac{c_1 c_3 \delta}{2c_2}\right)^{1/\beta}$.

This, together with the fact $\bar{\tau} \geq c\varepsilon$, implies that $|\bar{\tau}| D_{\bar{\tau}} u(X^*) \geq c\varepsilon\delta u(\underline{P_0})$. Using this in (4.6) we get

$$u(Y - \tau) \leq (1 - c\varepsilon\delta) u(\underline{P_0}).$$

Since y is any point in $B_\varepsilon(\underline{P_0})$, we obtain the the desired 'gap' with $\sigma = c\delta$, which is the desired conclusion. ■

We now propagate the 'gap' at the point $\underline{P_0}$ in the above inequality to a smaller gap in a whole neighborhood.

Lemma 10 *Let u be as Lemma 9, monotone increasing in every direction in $\Gamma'(e_n, \theta)$. Suppose for $\varepsilon > 0$, $\sigma > 0$ small we have*

$$u(\underline{P_0}) - v_\varepsilon(\underline{P_0}) \geq \sigma\varepsilon u(\underline{P_0}). \tag{3.10}$$

Then there exists positive constants C and h such that in Ψ we have

$$u(X) - v_{(1+h\sigma)\varepsilon}(X) \geq C\sigma\varepsilon u(\underline{P_0}).$$

Proof Write $v_\varepsilon(X) = \sup_{B'_\varepsilon(X)} u_1$ where $u_1(X) = u(X - \tau)$.

Let $\tau \in \Gamma'(e_n, \theta/2)$, with $\varepsilon = |\tau| \sin\frac{\theta}{2}$. For any unit vector ν (either in space or in $e_n - e_t$ plane depending on whether $\tau \in \Gamma_x$ or Γ_t) write

$$\begin{aligned} u(P) - & u_1(P + \varepsilon\nu(1 + h\sigma)) \\ &= [u(P) - u_1(P + \varepsilon\nu)] + [u_1(P + \varepsilon\nu) - u_1(P + \varepsilon\nu(1 + h\sigma))] \\ &= W(P) + Y(P). \end{aligned}$$

Set $\bar{\tau} = \tau - \varepsilon\nu$. Then $|\bar{\tau}| \geq |\tau| - \varepsilon \geq c\varepsilon$. We estimate $W(P)$ and $Y(P)$ as follows:

$W(P)$ is non-negative (since $\bar{\tau} \in \Gamma(e_n, \theta)$) and a solution to a parabolic equation, hence we can apply the Harnack and conclude

$$W(P) \geq cW(\underline{P_0}) \geq c\sigma\varepsilon u(\underline{P_0})$$

using our initial assumption (3.10).

For the $Y(P)$ term we apply the fact that $\nabla u \sim \frac{u}{d}$ and the Carleson Estimate. Hence

$$|\nabla u_1(P)| \leq Cu_1(P) \leq Cu_1(\underline{P_0}) \leq Cu(\underline{P_0}).$$

Here we have used a combination of the Carleson Estimate and the Backward Harnack Inequality to obtain the middle inequality.

Together the estimates for $W(P)$ and $Y(P)$ yield

$$W(P) + Y(P) \geq c\sigma\varepsilon u(\underline{P_0}) - Ch\sigma\varepsilon u(\underline{P_0}) \geq \bar{C}\sigma\varepsilon u(\underline{P_0})$$

if h is chosen small enough ($h < \frac{c}{2C}$).

■

Using the Backward Harnack Inequality we have the following corollary.

Corollary 2 *Let u be as in Lemma 10, monotone increasing in every direction in $\Gamma'(e_n, \theta)$. Suppose for $\varepsilon > 0$, $\sigma > 0$ small we have*

$$u(\underline{P_0}) - v_\varepsilon(\underline{P_0}) \geq \sigma\varepsilon u(\underline{P_0}).$$

Then there exists positive constants C and h such that in Ψ we have

$$u(X) - v_{(1+h\sigma)\varepsilon}(X) \geq C\sigma\varepsilon u(P_0).$$

An application of the geometric cone enlargement lemma due to Caffarelli, Theorem 4.2 in [CS] yields an expansion of the monotonicity cone, either Γ_t or Γ_x, as the case may be. This is stated precisely below.

Corollary 3 *Let u be a solution to our free boundary problem*

$$\begin{cases} \mathcal{L}u - u_t = 0 \quad in \;\; \{u > 0\} \cup \{u < 0\} \\ G(u_\nu^+, u_\nu^-) = 1 \quad along \;\; \partial\{u > 0\} \end{cases}$$

and set $u_r = \frac{u(rx, r^2 t)}{r}$ a parabolic blow-up. Then there exists an r_0 such that if $r \leq r_0$ we have the following:

1. *If u_r is monotone in a spacial cone of directions $\Gamma_0^x(e_n, \theta_0^x)$ then in Ψ u_r is monotone in an expanded cone of spacial directions $\Gamma_1^x(\nu_1, \theta_1^x)$ with defect angle decay given by $\delta_1 \leq c\delta_0$ and $|\nu_1 - e_n| \leq C\delta_0$.*

2. *If u_r is monotone in a space-time cone of direction $\Gamma_0^t(\eta_0, \theta_0^t)$, $\eta_0 \in \operatorname{span}\{e_n, e_t\}$, then in Ψ it is monotone in an expanded cone of directions $\Gamma_1^t(\eta_1, \theta_1^t)$ with defect angle decay $\mu_1 \leq c\mu_0$, $\eta_1 \in \operatorname{span}\{e_n, e_t\}$, $|\eta_1 - \eta_0| \leq c\mu_0$.*

3.4 Propagation Lemma

It is not possible to propagate the uniform gain in the monotonicity cone proved in the previous section to the free boundary. Instead only a portion of the gain can be propagated. This is accomplished by using a family of sup-convolutions with a variable radius.

For a positive function ϕ and direction τ define the sup-convolution

$$v_{\phi,\tau}(p) = \sup_{B_{\phi(p)}} u(q - \tau).$$

Also, let $\mathcal{C}_{R,T} = B'_R \times (-T, T)$.

In the sequel, we will need suitable versions of Lemmas 3.1 & 3.3 in [FS2]. The first describes a condition that ϕ needs to satisfy in order to make $v_{\phi,\tau}$ a sub/super-solution to our operator. The second establishes the existence of a family of functions satisfying this condition, among others.

Lemma 11 *Let u be a solution to our free boundary problem for the operator $\mathcal{L} - D_t$. Let ε_0 be small enough and $\phi \in C^2(\bar{\mathcal{C}}_{R,T})$ be a strictly positive function. Let $\omega \leq \omega(\phi_{\text{MAX}})$. Assume that in a smaller cylinder $\mathcal{C}' \subset \mathcal{C}_{R,T}$ with $\operatorname{dist}(\mathcal{C}', \partial\mathcal{C}_{R,T}) \geq \rho \gg \varepsilon_0$ $D_t\phi \geq 0$ and*

$$\mathcal{L}(\phi) - c_1 D_t\phi \geq C\frac{|\nabla\phi|^2 + \omega^2}{\phi} + c_2(|\nabla\phi| + \omega)$$

for some positive constants C_0, C, c_1 and c_2 depending only on $n, \lambda, \Lambda, \rho$.

Then in both $\Omega^{\pm}(v_{\psi,\tau}) \cap \mathcal{C}'$, $v_{\phi,\tau}$ is a viscosity subsolution to the operator $\mathcal{L} - D_t$.

Remark: In [FS2] this lemma is stated for a family of operators and is therefore slightly different, and more complex, than the version we have stated. We do not require this in our case. Additionally, in [FS2], the lemma is stated for a solution u to the Stefan problem, but the free boundary condition does not play a role in the proof and therefore the same result holds for our problem. Finally, their lemma has the Pucci extremal operator $\mathcal{P}^-$ on the left of the inequality instead of $\mathcal{L}$. Since $\mathcal{P}^-(\phi) - c_1 D_t\phi \leq \mathcal{L}(\phi) - c_1 D_t \phi$ by the properties of the Pucci operator, we are justified in making this substitution.

Next define the region

$$D = \left[B_1' \setminus \left(\bar{B}_{1/8}'(x_0)\right)\right] \times (-T, T).$$

From Lemma 3.3 [FS2] we have the following:

Lemma 12 *Let $T > 0$ and $C > 1$. There exists positive constants $\bar{C} = \bar{C}(T, C)$, $k = k(T, C)$, and $h_0' = h_0(T, C)$ such that for any $0 < h' < h_0'$ there is a family of C^2 functions ϕ_η, $0 \leq \eta \leq 1$, defined in the closure of D such that*

1. $1 - \omega \leq \phi_\eta \leq 1 + \eta h'$
2. $\mathcal{L}\phi_\eta - c_1 D_t \phi_\eta - C\frac{|\nabla\phi|^2 + \omega^2}{\phi_\eta} - c_2(|\nabla\phi| + \omega) \geq 0$ *in* D
3. $\phi_\eta \geq 1 + k\eta h'$ *in* $B_{1/2}' \times (\frac{-T}{2}, \frac{T}{2})$
4. $\phi_\eta \leq 1$ *in* $D \setminus (B_{7/8}' \times (-\frac{7T}{8}, \frac{7T}{8}))$
5. $D_t\phi_\eta, \leq \bar{C}\eta h'$ *and* $|\nabla\phi_\eta| \leq \bar{C}(\eta h' + \omega)$ *in* $\bar{D}$
6. $D_t \phi_\eta \geq 0$ *in* D

Here the ω appearing in (2) is a small positive constant; that is to say that if c_1, c_2, and ω are small positive constants depending on n, λ, Λ, C then it is possible to construct this family.

Remark: This is essentially Lemma 3.3 in [FS2] with only two small differences. First, as noted above, our domain has only the one hole; this causes only small and obvious alterations to the construction. Second, similar to the previous lemma, Lemma 3.3 in [FS2] has the Pucci extremal operator $\mathcal{P}_-$ instead of $\mathcal{L}$ in item (2). As in the previous remark, we are justified in this substitution by the properties of $\mathcal{P}^-$.

This concludes the essential properties of the variable radii functions ϕ_η.

In what follows we will use the family $\varepsilon\phi_{\sigma\eta}$. The $\sigma\eta$ term presents no difficulties but the derivative inequality which must be satisfied in order for the sup-convolutions to be subsolutions is not homogeneous in ϕ. Precisely, when we replace ϕ with $\varepsilon\phi$ in item (2) of the above lemma we have

$$\mathcal{L}\varepsilon\phi_\eta - c_1 D_t\varepsilon\phi_\eta - C\frac{|\nabla\varepsilon\phi|^2+\omega^2}{\varepsilon\phi_\eta} - c_2(|\nabla\varepsilon\phi| + \omega).$$

The presence of the ω terms prevents us from simply factoring out an ε. Rearranging we have

$$\left(\mathcal{L}\varepsilon\phi_\eta - c_1 D_t\varepsilon\phi_\eta - C\frac{|\nabla\varepsilon\phi_\eta|^2}{\varepsilon\phi_\eta} - c_2|\nabla\varepsilon\phi_\eta|\right) - C\frac{\omega^2}{\varepsilon\phi_\eta} - c_2\omega.$$

Owing to the condition initially satisfied by ϕ_η, the term in parentheses will be strictly positive provided ω is strictly positive (which it will be for a variable coefficient problem). So if we place the additional condition $\omega \leq \varepsilon^2$ then for sufficently small ε the family $\varepsilon\phi_{\sigma\eta}$ will satisfy the desired inequality.

This condition is what restricts us to using ε-monotonicity. In our later work we take $\varepsilon = |\tau|\sin\delta$, τ and δ coming from the monotonicity cone. Since we are also requiring $\omega \leq \varepsilon^2$ we see that some restriction on the length of τ is necessary. We cannot take τ to be arbitrarily small since that would in turn force the oscillation of of coefficient matrix, which is measured by ω, to be zero reducing the problem to the constant coefficient case.

We now prove our version of the propagation lemma used in this problem. From now on we will assume that $\omega_0 \le \varepsilon^2$ with $\varepsilon \le \varepsilon_0$. We will make use of standard asymptotic development results for both u and v_ε as described in Volume 2.

Lemma 13 *Let u_1 and u_2 be two viscosity solutions to our problem in $B_2' \times (-2, 2)$ and $F(u_2)$ Lipschitz continuous with $(0, 0) \in F(u_2)$. Assume*

1. *In $B_1' \times (-T, T)$*

$$v_\varepsilon(x, t) = \sup_{B_\varepsilon(x,t)} u_1 \le u_2(x, t)$$

2. *For some σ positive and some h small and $(x, t) \in B_{1/8}(x_0) \times (-T, T) \subset \{u_2 > 0\}$*

$$u_2(x, t) - v_{(1+h\sigma)\varepsilon}(x, t) \ge C\sigma\varepsilon u_2(x_0, 0)$$

3. *ω_0 is sufficiently small (as above).*

Then if $\varepsilon > 0$ and $h > 0$ are small enough, there exists $k \in (0, 1)$ such that in $B_{1/2} \times (-\frac{T}{2}, \frac{T}{2})$

$$v_{(1+kh\sigma)\varepsilon}(x, t) \le u_2(x, t)$$

Proof Define $w(x, t)$ as follows:

$$\begin{aligned} \mathcal{L}w - w_t = 0 \quad & \text{in } D \cap \{u_2 > 0\} \\ w = u(P_0) \quad & \text{on } \partial B_{1/8}(P_0) \times (-9T/10, 9T/10) \\ w = 0 \quad & \text{on the rest of } \partial_p D. \end{aligned}$$

Next, using the family constructed above with $\varepsilon \le \varepsilon_0$, set

$$\begin{aligned} v_\eta &= v_{\varepsilon\phi_{\sigma\eta}} \\ \bar{v}_\eta &= v_\eta + c\sigma\varepsilon w(x, t). \end{aligned}$$

The constant c is chosen to make $\bar{v}_\eta \le u_2$ on $\partial_p[B_{1/8}'(P_0) \times (-9T/10, 9T/10)]$. This is possible by the second hypothesis and the Harnack inequality. This ensures that $\bar{v}_0 \le u_2$.

We now demonstrate that the set of η for which $\bar{v}_\eta \le u_2$ is all of $[0,1]$. This is accomplished by showing that the set of η for which we have $\{\bar{v}_\eta > 0\} \cap D \Subset \{u_2 > 0\} \cap D$ is both open and closed; by construction the set is non-empty. The set is closed since the quantities involved vary continuously. We show it is open by supposing that there is an η for which the free boundaries touch, that is $\bar{v}_\eta(x_0,t_0) = u_2(x_0,t_0) = 0$.

All points are regular from the right for $\bar{v}_\eta$ by properties of the sup-convolution. Since $\bar{v}_\eta$ touches u_2 at (x_0,t_0), this point will be right regular for u_2. Additionally, by the assumption that $\bar{v}_\eta(x_0,t_0) = u_2(x_0,t_0) = 0$, we have that $v_\eta(x_0,t_0) = 0$ as well since w vanishes where u_2 does. This means that the corresponding point (y_0,s_0) on the free boundary of u_1 is left regular. Therefore, appealing to the asymptotic development results we have

$$
\begin{aligned}
&u_1 \ge a_1^+\langle y-y_0,\nu_1\rangle - a_1^-\langle y-y_0,\nu_1\rangle + o(|y-y_0|)\\
&\quad \text{with } G(a_1^+,a_1^-) \ge 1 \text{ and equality along } t = -\gamma\langle y-y_0,\nu_1\rangle\gamma > 0\\
&u_2 \le a_2^+\langle x-x_0,\nu_2\rangle - a_2^-\langle x-x_0,\nu_2\rangle + o(|x-x_0|)\\
&\quad \text{with } G(a_2^+,a_2^-) \le 1 \text{ and equality along } t = -\gamma\langle x-x_0,\nu_1\rangle, \gamma > 0\\
&v_\eta \ge a^+\langle x-x_0,\nu^*\rangle - a^-\langle x-x_0,\nu^*\rangle + o(|x-x_0|)
\end{aligned}
$$

where $\nu_1 = \frac{y_0-x_0}{|y_0-x_0|}$, $a^\pm = a_1^\pm|\tau|$, $\nu^* = \frac{\tau}{|\tau|}$ with

$$
\tau = \nu_1 + \frac{\varepsilon^2\phi_{\sigma\eta}(x_0,t_0)}{|y_0-x_0|}\nabla_x\phi(x_0,t_0).
$$

Now by the boundary Harnack comparison theorem we have $\frac{w}{u_2} \sim c$, so w has the asymptotic development ca_2^+. Hence for $\bar{v}_\eta$ we have

$$
\bar{v}_\eta \ge \bar{a}^+\langle x-x_0,\nu^*\rangle - a^-\langle x-x_0,\nu^*\rangle + o(|x-x_0|),
$$

where $\bar{a}^+ = a^+ + c\sigma\varepsilon a_2^+$. Now recall that G is Lipschitz continuous in both variables with Lipschitz constant L_G, increasing in the first, decreasing in the second. Moreover in [CS 9.14] it is shown that

$$\begin{aligned}|a_1^\pm - a^\pm| &\leq c(D_t\varepsilon\phi_{\sigma\eta}(x_0,t_0) + |\nabla\varepsilon\phi_{\sigma\eta}(x_0,t_0)|)\\ &\leq c(C\sigma\eta h\varepsilon + C\sigma\eta h\varepsilon + C\omega\varepsilon)\\ &\leq \bar{c}\sigma h\varepsilon,\end{aligned}$$

the last inequality coming from the construction of the ϕ. Specifically, we use the fact that $\eta \leq 1$ and $\omega \leq \varepsilon^2$ so the $C\omega\varepsilon$ term can be majorized by the linear term for small ε. Now $\bar{a}^+ \geq a_1^+ - \bar{c}\sigma\varepsilon h + c\sigma\varepsilon a_2^+$ and $a^- \leq a_1^- + \bar{c}\sigma\varepsilon h$. Hence we have

$$\begin{aligned}G(\bar{a}^+, a^-) &\geq G(a_1^+ - \bar{c}\sigma\varepsilon h + c\sigma\varepsilon a_2^+, a_1^- + \bar{c}\sigma\varepsilon h)\\ &\geq G(a_1^+, a_1^-) + L_G[(-\bar{c}\sigma\varepsilon h + c\sigma\varepsilon a_2^+) - \bar{c}\sigma\varepsilon h]\\ &= G(a_1^+, a_1^-) + L_G\sigma\varepsilon(-2\bar{c}h + ca_2^+)\\ &\geq 1 + L_G\sigma\varepsilon(-2\bar{c}h + ca_2^+),\end{aligned}$$

which implies that $G(a^+, a^-) > 1$ provided $h \leq \frac{ca_2^+}{4\bar{c}}$. Our non-degeneracy condition forces $a_2^+ \geq c > 0$, so taking $h = \frac{ca_2^+}{4\bar{c}}$ we will have $(-2\bar{c}h + ca_2^+) > 0$ and thus $G(a^+, a^-) > 1$ as desired.

We finish the proof by appealing to the Hopf Principle. The difference $u_2 - \bar{v}_\eta$ is a positive $\mathcal{L}$-supersolution in $\{\bar{v}_\eta > 0\}$ vanishing at the boundary point (x_0, t_0). This implies that $a_2^- \leq a^-$ and by the Hopf Principle we have $a_2^+ > \bar{a}^+$. The properties of G then imply that

$$1 \geq G(a_2^+, a_2^-) > G(\bar{a}^+, a^-),$$

which contradicts $G(\bar{a}^+, a^-) > 1$ above.

Now recalling the properties of the φ_η above, particularly

$$\varphi_\eta \geq 1 + k\eta h \text{ in } B'_{1/2} \times \left(\frac{-T}{2}, \frac{T}{2}\right)$$

we have for $\eta = 1$ $\varphi_\sigma \geq 1 + k\sigma h$ and thus

$$v_{\varepsilon(1+k\sigma h)}(x,t) \leq u_2(x,t)$$

in the region $B'_{1/2} \times \left(\frac{-T}{2}, \frac{T}{2}\right)$. ■

3.5 Spacial Regularity

3.5.1 Outline of Proof

In the constant coefficient case regularity follows from applying the interior gain, then the propagation lemma, then rescaling and repeating.

Preventing us from applying this classical argument in our case is the extra $\omega_0 \leq \varepsilon^2$ hypothesis of our propagation lemma. This restricts our choice of τ for which we can apply the propagation lemma with $u_1 = u(x - \tau)$ and $u_2 = u(x)$. The τ cannot be 'too short', since if it is allowed to be arbitrarily short it forces the oscillation $\omega_0 = 0$. This means that we can carry fully monotonicity using the propagation lemma only in the constant coefficient case. This forces us to use ε-monotonicity in our variable coefficient problem.

The reader will recall that a function u is ε_0-monotone in a unit direction τ if

$$u(x) \geq u(x - \varepsilon\tau) \quad \text{for } \varepsilon \geq \varepsilon_0.$$

Strict ε-monotonicity, which is of importance in this problem, is similar but quantifies the 'gap' between the two points:

$$u(x) - u(x - \varepsilon\tau) \geq c\varepsilon^\beta u(x) \quad \text{for } \varepsilon \geq \varepsilon_0, \text{ some } \beta > 0.$$

Clearly if u is fully monotone in a direction, then it is also ε_0-monotone for any ε_0 we choose.

Finally, it will be convenient to work with an alternate definition of ε-monotonicity, which is essentially equivalent to the one above. We say that u is ε-monotone in the cone of directions $\Gamma(\nu, \theta)$ with defect angle δ if for any $\tau \in \Gamma(\nu, \theta - \delta)$ with $|\tau| = \varepsilon$ we have

$$\sup_{B_{\varepsilon \sin \delta}(p)} u(q - \tau) \leq u(p).$$

In this case, the requirement of the propagation lemma is seen to be $\omega \leq (|\tau| \sin \delta)^2$.

Our method of proof modifies the classical proof by accommodating this ε-monotonicity. An outline of the steps involved is as follows: Interior gain (given by Corollary 3) is propagated to the free boundary by Lemma 13, but only for ε-monotonicity. The solution is then rescaled and by giving up part of the gain from the first two steps we can assert that the rescaled solution is fully monotone in a smaller cone away from the free boundary (see Lemma 17). This is all that is required to repeat the interior gain argument and at this point we can iterate the result. Special attention must be paid to the effect rescaling has on ε-monotonicity, as well as the amount of cone loss that occurs when in passing from ε-monotonicity to full monotonicity.

3.5.2 Spacial Cone Enlargement

In the lemma below, r and λ are constants (less than 1) chosen small enough later. In particular, λ will be chosen by the calculation in Corollary 5. We take $\varepsilon_k = \lambda^k \varepsilon_0$ and $C_{r^k} = B_{r^k/2} \times (\frac{-r^{2k}T}{2}, \frac{r^{2k}T}{2})$. Q_R will be the quadratic cylinder $B_R \times (-R^2, R^2)$.

Lemma 14 *Let u be a solution to our problem in $B_2' \times (-2, 2)$, monotone in the directions $\Gamma^x(e_n, \theta_0) \cup \Gamma^t(\eta, \theta_t)$, with η in the span of e_n and e_t. Then u is $r^k\varepsilon_k$-monotone in C_{r^k} in an expanded spacial cone of directions $\Gamma^x(\nu_1, \theta_1^x) = \Gamma_1^x$ with defect angle $\delta_1 \leq c\delta_0$, $c < 1$.*

Remark: Notice that we are asserting improved ε-monotonicity in smaller and smaller regions C_{r^k}, in *the same* expanded cone of directions Γ_1^x. Increasing the cone opening iteratively will come later.

Proof We rescale u:

$$u_r = \frac{u(rx, r^2t)}{r},$$

the rescaling factor r to be fixed later in the proof. The rescaled function will still possess the same spacial monotonicity cone as the original. Additionally, it solves an equation with the rescaled coefficients $a_{ij}(rx, r^2t)$. The oscillation of these coefficients in controlled by cr^{α}, α being the Hölder exponent. We will assume r is small enough so that Corollary 3 and the related results from Section 3 can be applied to u_r.

Consider now a spacial vector $\tau \in \Gamma^x(e_n, \theta - \delta_0)$, where δ_0 is the defect angle of the space cone, $|\tau| = \varepsilon \ll \delta_0$, $\bar{\varepsilon} = |\tau| \sin \delta_0$. Set $u_1(x,t) = u_r((x,t) - \tau)$. Additionally, assume that the defect angle of the space-time cone is less than that of the space cone.

From the monotonicity cone we have

$$\sup_{B_{\bar{\varepsilon}(x)}} u_1(y,t) \le u_r(x,t) \quad \text{in } B_1 \times (-1,1).$$

Note that this sup is performed over a space ball. However, we may assume that the same sort of result holds over a space-time ball

$$\sup_{B_{\bar{\varepsilon}(x,t)}} u_1(y,s) \le u_r(x,t) \quad \text{in } B_1 \times (-1,1) \tag{3.11}$$

since the defect angle in space is larger than that in time.

From Corollary 3 we have that there exists an enlarged cone of spacial directions $\tilde{\Gamma}_x$ in Ψ, the neighborhood of $(x_0, 0)$. Let $\bar{\tau}$ be a unit (spacial) direction in this expanded cone $\tilde{\Gamma}_x$; then since this enlarged cone contains the old one we can write this direction as $\bar{\tau} = \alpha\tau - \beta e_n$, $\beta \ge 0$, where τ is a unit vector in the old cone.

Since $\bar{\tau}$ is a direction in which u is increasing we have $D_{\bar{\tau}} u_r \ge 0$. Using the above, this implies that

$$D_\tau u_r \ge \frac{\beta}{\alpha} D_n u_r.$$

Now, if we delete a small neighborhood $\mathcal{N}$ of the contact line $\Gamma \cap \tilde{\Gamma}_x$ between the old and new cones, we can force $\frac{\beta}{\alpha} \ge c\delta_0$, with c depending on the size of the neighborhood $\mathcal{N}$ (see [CS], Section 9.4). We then obtain that for $\tau \in \Gamma_x \setminus \mathcal{N}$ we have

$$D_\tau u_r \ge c\delta_0 D_n u_r.$$

We now demonstrate that a similar inequality holds in this region if we allow the direction to have a small time component of order δ_0.

Let ω_1 and ω_2 be positive constants such that $\omega_1^2 + \omega_2^2 = 1$ and $|\omega_2| \leq \frac{c\delta_0}{2\bar{c}}$ (here $\omega_1 > 1/2$) where $\bar{c}$ is such that $|D_t u| \leq \bar{c} D_n u_r$ (this inequality is a consequence of the monotonicity cone). We then have

$$\omega_1 D_\tau u_r + \omega_2 D_t u_r \geq (\omega_1 c\delta_0 - \frac{c\delta_0}{2}) D_n u_r \geq \bar{c}\delta_0 D_n u_r.$$

Now if $\bar{\tau} = \tau + \bar{\varepsilon}\varrho$, where ϱ can be any $(n+1)-$dimensional unit vector, we have

$$u_r((x,t) - \bar{\tau}) - u_r(x,t) = -D_{\bar{\tau}} u_r(\tilde{x},\tilde{t})|\bar{\tau}| \leq -c\bar{\varepsilon}\delta_0 D_n u_r(\tilde{x},\tilde{t}) \leq -c\bar{\varepsilon}\delta_0 u_r(x_0,0).$$

In the last inequality we have used that $|\bar{\tau}| \geq c\varepsilon \geq c\bar{\varepsilon}$, $D_n u_r \sim \frac{u_r}{d}$ and the Harnack Inequalities. Note that $u_r((x,t) - \bar{\tau}) = u_r((x,t) - \tau - \bar{\varepsilon}\varrho)$ so as ϱ varies we obtain in this region

$$v_{\bar{\varepsilon}}(x,t) = \sup_{B_{\bar{\varepsilon}}(x,t)} u_1 \leq u_r(x,t) - c\bar{\varepsilon}\delta_0 u_r(x_0,0).$$

Now by standard arguments, as in Section 4, this gap implies that there exists a small h such that in Ψ (with a different constant c)

$$u_r(x,t) - v_{(1+h\delta)\bar{\varepsilon}}(x,t) \geq c\bar{\varepsilon}\delta_0 u_r(x_0,0).$$

At this point we must restrict ourselves ε-monotonicity so that the propagation lemma can be applied. Select τ with $|\tau| = \lambda\varepsilon_0 = \varepsilon_1$ and take r small enough so that $cr^\alpha \leq \varepsilon_1^2$ holds.

Now cr^α is the oscillation of the coefficient matrix A. This choice enables us to apply our propagation lemma and obtain that in $B_{1/2} \times (\frac{-T}{2}, \frac{T}{2})$ (here T is the 'height' of the cylinder Ψ)

$$u_r(x,t) \geq v_{(1+ch\delta_0)\bar{\varepsilon}_1}(x,t).$$

Therefore u_r is ε_1-monotone in an enlarged cone $\Gamma(\nu_1, \theta_1)$ in the region $B_{1/2} \times (\frac{-T}{2}, \frac{T}{2})$ with defect angle $\delta_1 = \frac{\pi}{2} - \theta_1 \leq c\delta_0$ with $c < 1$. Back-scaling we obtain $r\varepsilon_1$-monotonicity for u in the appropriately rescaled domain $C_r = B_{r/2} \times (\frac{-r^2 T}{2}, \frac{r^2 T}{2})$

Now we can repeat this argument for $\varepsilon_k = \lambda^k \varepsilon_0$ and r^k, λ to be chosen later. Precisely, u is fully monotone in the original cone, so u_{r^k} will be fully monotone in the original cone as well. Hence, it is ε_k-monotone no matter what we choose λ to be. Additionally, parabolic blowups decrease the defect angle of the space-time cone so we have that (3.11) will hold for any r^k.

We can repeat the cone enlargement arguments away from the free boundary to enlarge the spacial monotonicity cone. Finally, we can use the propagation lemma to transfer a portion of this new cone to the free boundary provided that $\omega_k \leq (\varepsilon_k)^2$. Since $\omega_k = cr^{\alpha k}$, we require that at each step $cr^{\alpha k} \leq \lambda^{2k}\varepsilon_0^2$, which can be arranged by coupling the choice of r and λ.

This proves that in C_{r^k} u is $r^k\varepsilon_k$-monotone in the new, larger, cone of directions $\Gamma^x(\nu_1, \theta_1)$ (we get the same enlarged cone in each case). Alternatively u_{r^k} is ε_k-monotone in $B_{1/2} \times (-\frac{T}{2}, \frac{T}{2})$ in this same cone. ■

We now turn to the task of iteratively increasing the monotonicity cone in the above result. To do this we will need to know that our solutions are fully monotone away from the free boundary since our interior gain results rely on full monotonicity. This in turn requires a strict ε-monotonicity not present in the result above. A slight modification of the proof, however, will yield the desired result. We explicitly observe that the enlarged cone in Corollary 4 below is not the same as in Lemma 14.

In the sequel we will let γ, δ be positive constants such that

$$0 < \gamma = \frac{1-\delta}{2}, \quad 0 < \beta < \min\left\{\gamma, \frac{\alpha+\delta-1}{2}\right\}$$

Notice in particular that this choice implies $\alpha + \delta - 1 > 0$ and $\delta < 1$. This δ is not to be confused with the defect angles δ_k. The M in the corollary below is determined by Lemma 17 below.

Corollary 4 *Let u be a solution to our problem, monotone in the directions $\Gamma^x(e_n, \theta_0) \cup \Gamma^t(\eta, \theta_t)$, with η in the span of e_n and e_t. Then u is $r^k\varepsilon_k$-monotone in C_{r^k} in an expanded spacial cone of directions $\Gamma^x(\nu_1, \theta_1^x) = \Gamma_1^x$ with defect angle $\delta_1 \leq c\delta_0$, $c < 1$.*

Alternatively u_{r^k} is ε_k-monotone in $B_{1/2} \times (-\frac{T}{2}, \frac{T}{2})$ in this cone. Furthermore, there exists an M such that, $M\bar{\varepsilon}_k^{\gamma}$ away from the free boundary, we have strict ε_k-monotonicity in these directions in the following sense:

$$u_{r^k}(p) - u_{r^k}(p-\tau) \geq c\sigma\bar{\varepsilon}_k^{1-\gamma} u_{r^k}(p). \tag{3.12}$$

Proof We pick up the proof of the above lemma at the point where the propagation lemma is applied, slightly changing notation with $\sigma = c\delta_0$ and p and q space-time points, so that

$$\sup_{B_{(1+h\sigma)\bar{\varepsilon}}(p)} u(q-\tau) \leq u(p).$$

Reducing the radius of the ball to $(1+\frac{h\sigma}{2})\bar{\varepsilon}$ we have

$$\sup_{B_{(1+\frac{h}{2}\sigma)\bar{\varepsilon}}(p)} u(q-\tau) + \min_{B_{(1+h\sigma)\bar{\varepsilon}}(p)} |\nabla u| |\frac{h\sigma\bar{\varepsilon}}{2}| \leq \sup_{B_{(1+h\sigma)\bar{\varepsilon}}(p)} u(q-\tau).$$

Now assume that p is located a distance $M\bar{\varepsilon}^{\gamma}$ away from the free boundary with M a large constant to be fixed later (see Lemma 11). We have $|\nabla u| \sim \frac{u}{d}$, d being the distance to the free boundary, hence the minimum of $|\nabla u|$ can be compared to the minimum of $\frac{u}{d}$.

By the Harnack inequalities u is comparable to $u(p)$, while for the distance we have

$$M\bar{\varepsilon}^{\gamma} - (1+h\sigma)\bar{\varepsilon} \leq d \leq M\bar{\varepsilon}^{\gamma} + (1+h\sigma)\bar{\varepsilon}.$$

Since $1+h\sigma$ is bounded, this implies that d is also comparable to $M\bar{\varepsilon}^{\gamma}$ for $\bar{\varepsilon}$ sufficiently small.

In turn this implies that

$$\begin{aligned}
\sup_{B_{(1+\frac{h}{2}\sigma)\bar{\varepsilon}}(p)} u(q-\tau) &\leq \sup_{B_{(1+h\sigma)\bar{\varepsilon}}(p)} u(q-\tau) - \min_{B_{(1+h\sigma)\bar{\varepsilon}}(p)} |\nabla u| |\frac{h\sigma\bar{\varepsilon}}{2} \\
&\leq \sup_{B_{(1+h\sigma)\bar{\varepsilon}}(p)} u(q-\tau) - C\frac{u(p)}{M\bar{\varepsilon}^{\gamma}} h\sigma\bar{\varepsilon} \\
&\leq \sup_{B_{(1+h\sigma)\bar{\varepsilon}}(p)} u(q-\tau) - Cu(p)\sigma\bar{\varepsilon}^{1-\gamma} \\
&\leq u(p) - Cu(p)\sigma\bar{\varepsilon}^{1-\gamma}.
\end{aligned}$$

Thus, by reducing slightly the cone of monotonicity, strict monotonicity is obtained away from the free boundary. ∎

3.5.3 Results regarding ε-monotonicity

As mentioned above, we will need to know that our solutions enjoy full monotonicity away from the free boundary. The above corollary is the first step in this process. The remaining results have been collected in this section.

Set

$$Q_{\sqrt{\varepsilon M}}(x^*, t^*) = B'_{\sqrt{\varepsilon M}}(x^*) \times (-M\varepsilon + t^*, M\varepsilon + t^*) \subset \Omega^+(u) \cap \{d_{x,t} > (M\varepsilon)^\gamma\}$$

and let ζ be the solution to the Dirichlet Problem

$$\begin{aligned} \zeta_t &= \mathcal{L}_{p^*}\zeta \quad \text{in } Q_{\sqrt{\varepsilon M}}(p^*) \\ \zeta &= u \quad \text{on } \partial_p Q_{\sqrt{\varepsilon M}}(p^*) \end{aligned}$$

where $\mathcal{L}_{p*} = \sum a_{ij}(p^*)D_{ij}u$, a constant coefficient operator.

We state the following result from [FS1] regarding ζ. Recall that α is the Hölder exponent of our coefficients. The value of M is the lemma below is to be determined later Lemma 17.

Lemma 15 (2.5 in [FS1]) *Let u be our caloric function and ζ as above. Let $\alpha, \gamma, \delta > 0$ be as in [Lemma 2.4 in FS1]. Then for every point p^* outside a $(M\varepsilon)^\gamma$-neighborhood of the free boundary of u, and for every point $p \in Q_{\sqrt{\varepsilon M}}$ the following estimates hold:*

$$|u(p) - \zeta(p)| \leq C(M\varepsilon)^{\frac{\alpha}{2}+\delta} u(p^*) \tag{3.13}$$

$$|D_n u(p) - D_n\zeta(p)| \leq C(M\varepsilon)^{\frac{\alpha}{2}+\frac{\delta}{2}} D_n u(p^*) \tag{3.14}$$

$$|D_t u(p) - D_t\zeta(p)| \leq C(M\varepsilon)^{\frac{\alpha+\delta-1}{2}} D_n u(p^*). \tag{3.15}$$

This next lemma allows us to transfer ε-monotonicity from u to ζ provided we have strict monotonicity with the correct power.

Lemma 16 *Let u and ζ be as above and suppose that u is strictly ε-monotone in a direction τ in the following sense:*

$$u(p) - u(p - \varepsilon\tau) \geq c\varepsilon^{1-\gamma} u(p).$$

Then if ε is sufficiently small, ζ is ε monotone in the direction τ.

Proof Define

$$u_1(p) = u(p) - u(p - \varepsilon\tau), \quad \zeta_1(p) = \zeta(p) - \zeta(p - \varepsilon\tau)$$

Using the above estimate we have

$$\zeta_1(p) \geq u_(p) - C_0(M\varepsilon)^{\delta+\frac{\alpha}{2}} u(p) \tag{3.16}$$

$$\geq c(\varepsilon)^{1-\gamma} u(p) - C_0(M\varepsilon)^{\delta+\frac{\alpha}{2}} u(p) \tag{3.17}$$

$$\geq 0 \tag{3.18}$$

for ε small enough, provided $1 - \gamma < \frac{\alpha}{2} + \delta$. Since

$$\delta + \frac{\alpha}{2} = \left(\frac{\delta}{2} + \frac{\alpha}{2}\right) + \frac{\delta}{2} > \frac{1}{2} + \frac{\delta}{2} = 1 - \gamma,$$

we have the desired inequality and the proof is complete. ■

Remark: The strict ε-monotonicity our solutions will enjoy from the previous section is

$$u(p) - u(p - \varepsilon\tau) \geq c\sigma\bar{\varepsilon}^{1-\gamma} u(p) \geq c\delta_0^{2-\gamma} \varepsilon^{1-y} u(p),$$

where $\sigma = c\delta_0$ and in future iterations we will have $\sigma_k = c\delta_k = c\bar{c}^k\delta_0$. We will be interested in applying the above lemma to a solution which is ε_k-monotone, in which case the $\bar{\varepsilon}$ appearing in the above will be $\bar{\varepsilon}_k$. Now for a fixed value of σ, there exists an ε_0 such that $\zeta_1(p) \geq 0$ as in the proof above. This value of ε_0 will depend on the size of σ, which could be problematic since $\sigma = c\delta_0$, and the defect angle will go to zero in our iteration.

However, in our iteration we will eventually have $\sigma_k = c\delta_k = c\bar{c}^k\delta_0$ and $\bar{\varepsilon}_k$. By choosing $\bar{c}$ close to 1, we can ensure that the calculation (3.16) remains valid when applied with δ_k and ε_k since the $C_0(M\varepsilon)^{\delta+\frac{\alpha}{2}} u(p)$ term will also be decreasing.

We have then

Lemma 17 *Let u be as in Corollary 4. Then, if ε_k is small enough and M is large enough, $(M\bar{\varepsilon}_k)^\gamma$ away from the free boundary, u_{r^k} is fully monotone in the cone $\Gamma_1^x(\nu, \theta_1 - c_0\varepsilon_k^{\frac{\alpha+\delta}{2}})$, $c_0 > 1$.*

Proof From Corollary 4

$$u_{r^k}(p) - u_{r^k}(p-\tau) \geq c\sigma\bar{\varepsilon}_k^{1-\gamma}u_{r^k}(p),$$

for $\tau \in \Gamma_1^x(\nu, \theta_1), |\tau| = \varepsilon_k$, $M\bar{\varepsilon}_k^\gamma$ from the free boundary.

Applying Lemma 16 we conclude that ζ is also ε_k-monotone in the cone Γ_1^x away from the free boundary. Now ζ solves a constant coefficient parabolic equation which we may assume is the heat equation. From the proof of Lemma 13.23 in [CS] we conclude that ζ is fully monotone in the cone of directions $\Gamma_1^x(\nu, \theta_1 - c\varepsilon)$.

Note that this cone of monotonicity implies $|\nabla\zeta|$ is controlled (in this region) by $D_n\zeta$. This in turn implies our estimate for $|D_n u - D_n\zeta|$ extends to an estimate of $|D_e u - D_e\zeta|$, where e is any spacial vector.

Using this, we have for any direction $e \in \Gamma_1^x(\nu, \theta_1 - c\varepsilon)$

$$D_e u(p^*) \geq D_e\zeta(p^*) - c(M\varepsilon_k)^{\frac{\alpha+\delta}{2}}D_n u(p^*).$$

It follows then that u is fully monotone in the direction $\bar{e} = e + c(M\varepsilon_k)^{\frac{\alpha+\delta}{2}}\nu$.

Hence u is fully monotone in the cone $\Gamma_1^x(\nu, \theta - c\varepsilon_k - c(M\varepsilon_k)^{\frac{\alpha+\delta}{2}})$. Now at this point in the proof the size of M has already been determined by invoking Lemma 13.23 in [CS]. We may assume ε_k is small enough that we can majorize the loss term and have full monotonicity in $\Gamma_1^x(\nu, \theta_1 - c_0\varepsilon^{\frac{\alpha+\delta}{2}})$, with $c_0 > 1$. ∎

Remark Lemma 13.23 in [CS] is stated in a slightly different context from what we have here. Its proof requires a control $|\zeta_t| \leq c|\nabla\zeta|$ which holds since such an estimate holds for u and since we have the estimates between the derivatives of u and ζ. More precisely, we have

$$\begin{aligned}|D_t\zeta| &\leq |D_t\zeta - D_t u| + |D_t\zeta| \\ &\leq c(M\varepsilon)^{\frac{\alpha+\delta-1}{2}} D_n u(p^*).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}|D_n u(p^*)| &\leq |D_n u(p^*) - D_n\zeta| + |D_n\zeta| \\ &\leq c(M\varepsilon)^{\frac{\alpha+\delta}{2}} D_n u(p^*) + |D_n\zeta|.\end{aligned}$$

Or,

$$(1 - c(M\varepsilon)^{\frac{\alpha+\delta}{2}})D_n u(p^*) \leq |D_n\zeta|.$$

So if we take ε small enough, we have control of $D_n u(p^*)$ by $D_n\zeta$. Here we have not specified an argument for $D_n\zeta$ since the above estimate holds for any point in the neighborhood.

Taken together, these imply the control of $D_t\zeta$ by $D_n\zeta$ needed in the proof of Lemma 13.23 in [CS] (naturally control by D_n implies control by the full gradient).

Lastly, we quote Lemma 2.4 from [FS1] for the space-time cone.

Lemma 18 *(Lemma 2.4 in [FS1]).*
Let $\alpha \leq 1$ be the Hölder exponent of the a_{ij} and β, δ, γ as indicated above. Suppose $u \geq 0$ is monotone in the e_n direction and

$$u(p) - u(p - \varepsilon\tau) \geq c\varepsilon^{1-\gamma+\beta} u(p) = c\varepsilon^{\frac{\alpha+\delta}{2}-1} u(p)$$

for $d_p < \eta/4$ [this is the distance to the free boundary] where $\tau = \beta_1 e_n + \beta_2 e_t$ with $\beta_1 > 0$ $|\beta_2 \neq 0$ and $\beta_1^2 + \beta_2^2 = 1$. Then if $M = M(n, L)$ is large enough and ε is small enough, outside a $(M\varepsilon)^\gamma$-neighborhood of the free boundary we have

$$D_{t_\varepsilon} \geq 0$$

where $t_\varepsilon = \tau + c(M\varepsilon)^{(\alpha+\delta-1)/2} e_n$.

3.5.4 Regularity of the Free Boundary in Space

By combining the results from our cone enlargement lemma and the ε-monotonicity section we have the following corollary suitable for iteration.

Corollary 5 *Let u be a solution to our problem, monotone in the directions $\Gamma^x(e_n, \theta_0) \cup \Gamma^t(\eta, \theta_t)$ with η in the span of e_n and e_t. Then u is $r^k\varepsilon_k$-monotone in Q_{r^k} in an expanded spacial cone of directions $\Gamma^x(\nu_1, \theta_1^x) = \Gamma_1^x$ with defect angle $\delta_1 \leq c\delta_0$, $c < 1$.*

Alternatively u_{r^k} is ε_k-monotone in Q_1. Furthermore, there exists an M such that, $M\bar{\varepsilon}_k^{\gamma}$ away from the free boundary, we have have strict ε_k-monotonicity in these directions in the following sense:

$$u_{r^k}(p) - u_{r^k}(p - \tau) \geq c\sigma\bar{\varepsilon}_k^{1-\gamma} u_{r^k}(p). \tag{3.19}$$

Finally, in this region, at a distance greater than $M\bar{\varepsilon}_k^{\gamma}$ from the free boundary, u_{r^k} is fully monotone in a cone of directions $\bar{\Gamma}_1^x(\nu, \bar{\theta}_1)$ with $\bar{\delta}_1 \leq \bar{c}\delta_0$.

Proof This is Corollary 4 except for the last part about full monotonicity.

By Corollary 4 we have u_{r^k} ε_k-monotone in Q_1 in the cone of directions $\Gamma^x(\nu_1, \theta_1)$, with (3.19) holding for directions in this cone. From Lemma 17 u_{r^k} is therefore fully monotone in the cone

$$\Gamma^x(\nu_1, \theta - c_0\varepsilon_k^{\frac{\alpha+\delta}{2}}) := \bar{\Gamma}_1^x.$$

For notational convenience we will write B for the power $\frac{\alpha+\delta}{2}$. In terms of the spacial defect angles, we know that $\delta_1 \leq c\delta_0$ with $c < 1$. Let $\bar{\delta}_1$ denote the defect angle of the cone $\bar{\Gamma}_1^x$. It is readily seen that the worst case scenario occurs with ε_1. In this case we have

$$\bar{\delta}_1 = \delta_1 + c_0\varepsilon_1^B \leq c\delta_0 + c_0\varepsilon_1^B.$$

We desire to preserve the geometric decay of the defect angles so we want $\bar{\delta}_1 \leq \bar{c}\delta_0$ with $\bar{c} < 1$. So what we must prove is that there is an appropriate choice of λ in the definition of $\varepsilon_k = \lambda^k\varepsilon_0$ that makes this possible.

Now $\varepsilon_1 = \lambda\varepsilon_0$, so we need

$$c_0\lambda^B\varepsilon_0^B \leq c'\delta_0,$$

where c' is chosen so that $c + c' = \bar{c} < 1$ and $c > 2c'$. Now our starting assumption is that $\varepsilon_0 \ll \delta_0$ (and thus we can also assume $\varepsilon_0^B \le \delta_0$; note that $B > 1/2$) so it suffices to chose λ so that

$$\lambda^B \le \frac{c'}{c_0}.$$

This would suffice to give $\bar{\delta}_1 \le \bar{c}\delta_0$.

∎

The calculation in the above proof will be of interest to us when we iterate. In particular, we need to ensure that the choice of λ made in Corollary 5 will also work in the iteration, where we need $\delta_k = \bar{c}^k \delta_0$. We take care of this now with the following result about cones.

Lemma 19 $\Gamma_k = \Gamma(\nu_k, \theta_k)$ *is a sequence of cones,* $\Gamma_k \subset \Gamma_{k+1}$ *with defect angle* $\delta_k \le c^k \delta_0$*. Let* $\bar{\Gamma}_k = \Gamma_k(\nu_k, \theta_k - c_0 \varepsilon_k^B)$, $B = \frac{\alpha+\delta}{2}$*, with* $\varepsilon_k = \lambda^k \varepsilon_0$*, and defect angle* $\bar{\delta}_k$*. Then there exists a* $\bar{c} < 1$ *such that* $\bar{\delta}_k \le \bar{c}^k \delta_0$*.*

Proof We have

$$\begin{aligned}
\bar{\delta}_k &= \delta_k + c_0(\varepsilon_k)^B \\
&\le c\bar{\delta}_{k-1} + c_0(\lambda^k \varepsilon_0)^B \\
&\le (c\bar{c}^{k-1} + (c')^k)\delta_0.
\end{aligned}$$

We want this last term to be less than $\bar{c}^k \delta_0$ for a choice of $\bar{c}$ independent of k. From $\bar{c} = c + c'$ (referring to the constants in the proof of Corollary 5 above) and the binomial theorem we have

$$\begin{aligned}
(c\bar{c}^{k-1} + (c')^k) &= c \sum_{n=0}^{k-1} \binom{k-1}{n} c^{k-1-n} (c')^n + (c')^k \\
&= \sum_{n=0}^{k-1} \binom{k-1}{n} c^{k-n} (c')^n + (c')^k.
\end{aligned}$$

Whereas

$$(c + c')^k = \sum_{n=0}^{k} \binom{k}{n} c^{k-n} (c')^n.$$

Consider the term $n = 1$. We have that the first expression has the term $(k-1)c^{k-1}c'$ while the second has $kc^{k-1}c'$. Provided $c' < c$ we will have

$$(k-1)c^{k-1}c' + (c')^k < kc^{k-1}c' < (k-1)c^{k-1}c' + c^{k-1}c' = kc^{k-1}c'$$

from which it follows that

$$(c\bar{c}^{k-1} + (c')^k) < (c + c')^k.$$

Therefore, with $\bar{c} = c + c' < 1$, we will have $\bar{\delta}_k \leq \bar{c}^k \delta_0$ for any k. ■

Remark: The results of this section imply that if a solution v to our problem is strictly ε-monotone in the sense of (3.19) in a cone of directions Γ_1 with defect angle $\delta_1 \leq c\delta_0$, then there exists $\bar{\Gamma}_1$, $\Gamma_0 \subset \bar{\Gamma}_1 \subset \Gamma_1$, with v fully monotone away from the free boundary in $\bar{\Gamma}_1$, still preserving a decay of the defect angle $\bar{\delta}_1 \leq \bar{c}\delta_0$.

3.5.5 Final Spacial Iteration

We reach the main result of this section.

Corollary 6 *The free boundary is a $C^{1,\alpha}$ surface in space.*

Proof Combining Corollary 4 and the results in the ε-monotonicity section, we have that u_{r^k} is ε_k-monotone in $B_{1/2} \times (-\frac{T}{2}, \frac{T}{2})$ in the cone of directions $\bar{\Gamma}_1$ with $\bar{\delta}_1 \leq \bar{c}\delta_0$. Furthermore, we have in this region

$$\sup_{B_{\bar{\varepsilon}_k}(p)} u(q - \tau) \leq u(p)$$

for $\tau \in \bar{\Gamma}_1$, $|\tau| = \varepsilon_k$. Additionally, u_{r^k} will be fully monotone in this cone of directions away from the free boundary. We may assume that $T < 1$. Then the quadratic cylinder $Q_{T/2} \subset B_{1/2} \times (-\frac{T}{2}, \frac{T}{2})$.

This implies that $u_{r^2 T/2}$ is $\frac{2}{T}\varepsilon_2$-monotone in $\bar{\Gamma}_1$ in the above sup sense, in all of Q_1 and is a solution in the larger region Q_2 (a technicality needed for the propagation lemma). Furthermore, $u_{r^kT/2}$ is fully monotone in $\bar{\Gamma}_1$ away from the free boundary in the region Ψ by virtue of the results in Section 6.3.

We can then apply the proof of Lemma 14 and Corollary 4 to $u_{r^2T/2}$, concluding that in $B_{1/2} \times (-\frac{T}{2}, \frac{T}{2})$, $u_{r^2T/2}$ is $\frac{2}{T}\varepsilon_2$-monotone in an enlarged cone of directions $\bar{\Gamma}_2(\nu_2, \bar{\theta}_2)$ with $\bar{\delta}_2 \leq \bar{c}^2\delta_0$. As was the case for Lemma 14, we have the same conclusion for $u_{r^kT/2}$, $k \geq 2$. Note that this region contains $B_{T/2} \times (-\frac{T}{2}, \frac{T}{2})$.

Using this observation, after back-scaling $u_{r^2T/2}$ we deduce that u is $r^2\varepsilon_2$-monotone in a cone of directions $\bar{\Gamma}_2$ in the region $B_{r^2T^2/2^2} \times (-\frac{r^4T^3}{2^3}, \frac{r^4T^3}{2^3}) \supset Q_{r^2T^2/2^2}$.

In this way we construct a sequence of parabolic neighborhoods of the origin $Q_{r^kT^k/2^k}$ in which u is $r^k\varepsilon_k$-monotone in a cone of directions $\bar{\Gamma}_k$. This implies that the free boundary of u intersected with the time level $\{t = 0\}$ is a $C^{1,\alpha}$ surface in space due to the following calculus lemma.

■

Lemma 20 *Let f be a function defined in a region $\mathcal{D}$ monotone in contracting cylinders $C_k = B'_{R^k} \times (-b_k, b_k)$ in cones Γ_k, with defect angles $\delta_k \leq \lambda^k\delta_0$, $\lambda < 1$. Additionally, assume $f(0) = 0$. Then $\{f = 0\}$ is a $C^{1,\alpha}$ surface.*

Proof Since we can center the neighborhoods at any point on the free boundary, we have at once that each point on the free boundary possesses a genuine normal vector. It remains then only to show that these normal vectors vary with a modulus of continuity. It suffices for our purposes to assume the origin is one of the points, the other will be denoted by x; their corresponding normal vectors will be denoted by ν_x and ν_0.

Select k such that $R^{k+1} < |x| \leq R^k$ and let Γ_{k+1} and Γ_k be the corresponding monotonicity cones. Now the crucial observation is that the monotonicity cone is the same for any point in the corresponding region. In particular, both x and 0 have monotonicity cone Γ_k since they are both in the region C_k. In turn this implies that the distance between the normal vectors ν_0 and ν_x is controlled by the defect angle of the monotonicity cone:

$$|\nu_x - \nu_0| \leq 2\delta_k.$$

Now select $\alpha \in (0,1)$ such that $R^\alpha = \lambda$. Then we have

$$\begin{aligned} |\nu_x - \nu_0| \le 2\delta_k &= c\lambda^k \\ &= c(R^\alpha)^k \\ &= c\left(\frac{R^{k+1}}{R}\right)^\alpha \\ &\le C|x|^\alpha. \end{aligned}$$

■

3.6 Regularity of the Free Boundary in Space-Time

We will now use similar ideas to prove that the free boundary has a space-time normal at every point which varies with a Hölder modulus of continuity. When taken together with the spacial regularity proved in the previous section this result will complete the regularity of the free boundary.

Having proved spacial regularity in the previous section we can orient our problem so that e_n is the spacial normal at the origin. We will prove that there exists a space-time normal at the origin in the $e_n - e_t$ plane. This will be the full normal to the free boundary at that point.

The same technique used to prove the spacial regularity will be used for the space-time regularity. A technical difficulty arises early when following this line of argument however. Recall that in the spacial case we used the 'sup-convolution' concept to describe the monotonicity cone. Precisely, given any $\tau \in \Gamma(e_n, \theta - \delta)$, δ the spacial defect angle, we have that

$$\sup_{B'_\varepsilon(x)} u(y - \tau, t) \le u(x,t).$$

Here the B' denotes the ball in space only. We have that $\varepsilon = |\tau| \sin \delta$. However, we need to know that the same statement holds over a full space-time ball B_ε. Since parabolic rescalings depress the space-time defect angle μ we can assume that $\delta \ge \mu$

at every step in the iteration. This guarantees that the full ball B_ε is contained in the monotonicity cone.

In the present case however, the fact that the space-time defect angle μ is always smaller than the spacial defect angle δ poses a difficulty. It is still true that for $\tau \in \Gamma_t(\nu, \theta_t - \mu)$ we can take the sup statement over the 'thin' ball, this time in the $e_n - e_t$ plane, but it is no longer true that we can take the sup over the full ball of radius $|\tau| \sin \mu$.

We have the following technical geometric lemma to address this problem.

Lemma 21 *Let u be monotone in the directions $\Gamma_x(\theta_x) \cup \Gamma_t(\theta_t)$ with defect angles $\delta \geq \mu$ and $\delta \leq \pi/6$. Then there exists a $\kappa > 0$ and a μ_0 such that for any $\tau \in \Gamma_t(\theta_t - \kappa\mu)$ with $\mu \leq \mu_0$ the full ball B_ε centered at the endpoint of τ is contained in the monotonicity cone, with $\varepsilon = |\tau| \sin \kappa\mu$.*

Proof The space time cone is two dimensional in the plane $e_n - e_t$, while the spacial cone is a right cone in space. It therefore suffices to prove the result in three dimensions. Additionally, owing to the purely geometric nature of the lemma we may assume that the cones open along the positive z-axis. We will assume that the space-time cone opens along the y-axis.

Under these assumptions the elliptic monotonicity cone with defect angles δ and μ has parametric equations in the variables s, t

$$(\cot(\delta) s \cos t, \cot(\mu) s \sin t, s)$$

with $0 \leq t \leq 2\pi$, $s \geq 0$. The vector

$$v = (0, \cos \mu, \sin \mu)$$

is on this cone; it is one edge of space-time direction with unit length.

Define for $\mu^* = (1 + \kappa)\mu = c\mu$ the unit vector τ as

$$\tau = (0, \cos \mu^*, \sin \mu^*).$$

Then τ will be an edge of the smaller space-time cone $\Gamma(\theta_t - \kappa\mu)$.

We want to show that for some choice of κ no vector on the edge of the elliptic cone can make an angle with τ which is smaller than the angle τ makes with v. This would mean that the right cone of directions with axis τ and opening given by the angle between v and τ, which is $\kappa\mu$, would fit completely inside the elliptic cone. In turn this would imply that the ball with radius $|\tau| \sin \kappa\mu$ centered at the endpoint of τ is entirely contained in the elliptic cone, and this is the conclusion of the lemma.

Let $\alpha(\, , \,)$ denote the angle between two vectors and let

$$w = (\cot(\delta)x, \cot(\mu)y, 1)$$

with $x^2 + y^2 = 1$; any vector on the outer edge of the cone lies in the same direction as w for some such choice of x, y.

We want to show that for any such w

$$\alpha(\tau, v) \leq \alpha(w, \tau).$$

Or, since the cosine is decreasing in the first quadrant

$$\cos(\alpha(\tau, v)) \geq \cos(\alpha(w, \tau)).$$

We then use the characterization of the dot product to obtain (τ and v are unit vectors)

$$\tau \cdot v \geq \frac{\tau \cdot w}{|w|}.$$

We compute (using $x^2 + y^2 = 1$)

$$\cos\mu^* \cos\mu + \sin\mu \sin\mu^* \geq \frac{y \cos\mu^* \cot\mu + \sin\mu^*}{\sqrt{1 + (1 - y^2)\cot^2\delta + y^2 \cot^2\mu}}.$$

Now it can be shown directly from calculus that the right hand side as a function of y with all other variables fixed increases to a maximum at

$$y_M = \frac{\cot\mu \cos\mu^*(1 + \cot^2\delta)}{\sin\mu^*(\cot^2\mu - \cot^2\delta)}.$$

When $y = 1$ the two sides are equal so to obtain our desired inequality for all $0 \leq y \leq 1$ it is necessary for $y_M \geq 1$. Recall that $\mu^* = c\mu$, $c > 1$. We begin to estimate (we are assuming $\mu < \delta$ so $\cot \mu > \cot \delta$)

$$\begin{aligned} y_M &= \frac{\cot \mu \cos \mu^* (1 + \cot^2 \delta)}{\sin \mu^* (\cot^2 \mu - \cot^2 \delta)} \\ &\geq \frac{\cot \mu \cot \mu^* (1 + \cot^2 \delta)}{(\cot^2 \mu)} \\ &= (1 + \cot^2 \delta) \frac{\cot \mu^*}{\cot \mu} \\ &= (1 + \cot^2 \delta) \frac{\tan \mu}{\tan \mu^*} = (1 + \cot^2 \delta) \frac{\tan \mu}{\tan c\mu}. \end{aligned}$$

Letting $\mu \to 0$ in this last line we obtain by L'Hôpital's rule

$$(1 + \cot^2 \delta) \frac{1}{c}.$$

We need

$$(1 + \cot^2 \delta) \geq c > 1.$$

Or, to provide some room

$$(1 + \cot^2 \delta) \geq c > 2.$$

Now, by assumption, $\delta \geq \pi/6$, so that $(1 + \cot^2 \delta) > 2$. Thus, we can find a μ_0 such that for $\mu \leq \mu_0$ there exists a c for which

$$y_M \geq (1 + \cot^2 \delta) \frac{\tan \mu}{\tan c\mu} \geq 2.$$

In turn this implies that the ball centered at the tip of $\tau \in \Gamma_t(\theta_t - \kappa\mu)$ of radius $|\tau| \sin \kappa\mu$ will be completely contained in the monotonicity cone. ∎

Since we have already proved the spacial regularity of the solution and we know that parabolic rescalings depress the space-time defect angle μ we will assume throughout the rest of this section that the hypotheses of this lemma are satisfied.

Our proof of the space-time regularity can now proceed along the same lines as the spacial regularity. Namely, we prove an enlargement of the monotonicity cone away from the free boundary, transfer a portion of this enlarged cone to the free boundary and iterate via a parabolic rescaling. As in the spacial case care must be taken with the iteration necessary to accommodate working with ε-monotonicity rather than full monotonicity.

A technical complication not present in the spacial case is the fact that parabolic rescalings enlarge the space-time cone Γ_t. Indeed, if u has monotonicity cone Γ_t with defect angle μ then u_r, a parabolic rescaling of u, will have a monotonicity cone which has defect angle $r\mu$.

We note that this gain from rescaling is of no use to us in proving the enlargement of the monotonicity cone since any gain from the rescaling must be given back when the solution is back-scaled. We will write ${}^*\Gamma_0^t$ to mean the original cone Γ_0^t dilated by the rescaling. The rescaling factor involved in this dilation will be clear from the context so we will suppress it from the already dense notation.

We begin with a lemma which is the analogy of Lemma 14 and Corollary 4 for the space-time case and which largely follows the same lines for its proof. The only major difference is the need to keep track of the dilation of the space-time cone due to the rescaling. As before, M and $\varepsilon_k = \lambda^k \varepsilon_0$ for $\lambda < 1$ are chosen later. Additionally, although the statement of the lemma is similar to the spacial case we do not necessarily have that the constants, including r and λ, are the same.

Lemma 22 *Let u be a solution to our problem, monotone in the directions $\Gamma^x(e_n, \theta_0) \cup \Gamma^t(\eta, \theta_t)$ with η in the span of e_n and e_t. Then u is $r^k\varepsilon_k$-monotone in C_{r^k} in an expanded space-time cone of directions $\Gamma^t(\nu_1, \theta_1^t) = \Gamma_1^t$ with defect angle $\mu_1 \leq c\mu_0$, $c < 1$.*

Alternatively u_{r^k} is ε_k-monotone in $B_{1/2} \times (-\frac{T}{2}, \frac{T}{2})$ in the corresponding r^k-dilated cones ${}^*\Gamma_1^t$. Furthermore, there exists an M such that, $M\bar{\varepsilon}_k^{\gamma}$ *away from the free boundary, we have have strict ε_k-monotonicity in these directions in the following sense:*

$$u_{r^k}(x,t) - u_{r^k}((x,t) - \tau) \geq c\sigma\bar{\varepsilon}_k^{1-\gamma} u_{r^k}(x,t). \tag{3.20}$$

Proof As in the spacial case we begin with a rescaling; as in that proof the choice of r will be coupled to λ and chosen later.

$$u_r(x,t) = \frac{u(rx, r^2 t)}{r}.$$

For u_r its space-time cone ${}^*\Gamma_0^t$ is described as the cone in the the $e_n - e_t$ plane with edges $e_t + Be_n$, $-e_t - Ae_n$. From Corollary 3 we have that the space-time cone enlarges. As in the spacial case we have either (the situation is simpler in this case since the cone is only two-dimensional)

$$D_t u_r + BD_n u_r \geq cr\mu D_n u_r \quad \forall (x,t) \in \Psi$$

or

$$-D_t u_r - AD_n u_r \geq cr\mu D_n u_r \quad \forall (x,t) \in \Psi.$$

Recall that ${}^*\Gamma_0^t$ has defect angle $r\mu$; that is why an $r\mu$ appears on the right hand side.

We will assume that the first holds; the other case it treated similarly. For convenience we will denote by σ the direction $e_t + Be_n$. Let τ be the direction in the $e_n - e_t$ plane which lies below σ by the angle $\kappa r\mu$.

Then define

$$u_1(x,t) = u_r((x,t) - \tau).$$

Then we have that

$$\sup_{B_\varepsilon(x,t)} u_1(x,t) \leq u_r(x,t)$$

throughout the whole cylinder by our previous lemma where $\varepsilon = |\tau| \sin \kappa r\mu$.

Similar to the spacial case, the inequality

$$D_\tau u_r \geq cr\mu D_n u_r \quad \forall (x,t) \in \Psi$$

then holds. We needed the result about σ to know which direction the cone was increasing in; once we have this information we only need to work with τ.

Next, we show that a similar inequaltiy holds for small perturbations of this direction τ by other directions.

Let $\omega_1 > 1/2$ and ω_2 be such that $\omega_1^2 + \omega_2^2 = 1$. Next recall that for any spacial direction e we have $|D_e u_r| \leq c^* D_n u_r$ and a similar inequality holds for time derivatives by the monotonicity cone. Thus for $\varrho \in \mathbb{R}^{n+1}$

$$\omega_1 D_\tau u_r + \omega_2 D_\varrho u_r \geq (\omega_1 c\mu - \omega_2 c^*) D_n u_r \geq cr\mu D_n u_r$$

provided $|\omega_2| \leq \frac{cr\mu}{2c*}$.

Set $\bar{\tau} = \tau + \varepsilon\varrho$ so that the above implies

$$u_r((x,t) - \bar{\tau}) - u_r(x,t) = -D_{\bar{\tau}} u_r(\tilde{x}, \tilde{t})|\bar{\tau}| \leq -c\varepsilon r\mu D_n u(\tilde{x}, \tilde{t}) \leq -c\varepsilon r\mu u_r(x_0, 0).$$

As in the spacial case this inequality implies that

$$u_r((x,t) - \bar{\tau}) - u_r(x,t) = u_r((x,t) - \tau - \varepsilon\varrho) - u_r(x,t) \leq -c\varepsilon r\mu u_r(x_0, 0).$$

As ϱ ranges over all possible direction we deduce that in the region Ψ

$$v_\varepsilon(x,t) := \sup_{B_\varepsilon(x,t)} u_1(x,t) \leq u_r(x,t) - cr\mu u_r(x_0, 0).$$

This enlarges to yield that in Ψ for a small h we have

$$u_r(x,t) - v_{(1+h\mu)\varepsilon}(x,t) \geq c\varepsilon r\mu u_r(x_0, 0).$$

which is the cone enlargement of the cone ${}^*\Gamma_0^t$ away from the free boundary.

It is at this point we must once again restrict ourselves to ε-monotonicity so that the propagation lemma can be applied to u_r and u_1. The remainder of the proof then proceeds in the same fashion as that of the spacial case. ■

At this point the argument follows identical lines to that of the spacial case by using Lemma 18 to produce a slightly smaller cone of direction ${}^*\bar{\Gamma}_1^t$ in which the solution is fully monotone away from the free boundary; additionally as in the spacial case a careful choice of r and λ results in the cones ${}^*\bar{\Gamma}_1^t$ still preserving the decay of the defect with $\mu_1 \leq \bar{c}\mu_0$. An iteration argument then implies the following corollary.

Corollary 7 *The solution u is $r^k\varepsilon_k$ monotone in the parabolic neighborhoods of the origin $Q_{r^k T^k/2^k}$ in the cone of directions $\bar{\Gamma}_k^t$ which have defect angles $\mu_k \leq \bar{c}^k\mu$, $\bar{c} < 1$.*

We arrive at the proof of our main theorem:

Proof *(Theorem 2)*

The existence of a full normal at the origin follows by Corollary 7 and the spacial regularity proved in Corollary 6. By centering this argument at different points we obtain that a normal vector to the free boundary exists at every point of the free boundary in $Q_{1/2}$. Furthermore, the spacial part of this normal vector varies with a Hölder modulus of continuity and the iteration from Corollary 7 implies that the space-time part of this normal also varies with a Hölder modulus of continuity.

Together this implies both the existence of a normal vector $\eta(x,t)$ at each point on the free boundary and also that this normal vector varies with the moduli of continuity stated in Theorem 2.

As in the proof of the main result in [ACS3], to finish we apply the results in [W] to our solution now that we know the free boundary is $C^{1,\alpha}$ for each time level. This implies that $\nabla_x u$ is continuous up to the boundary at every time level. Hence u takes up its boundary condition with continuity and u is a classical solution to our problem. ∎

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